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Dynamic State Estimation for Radial Microgrid Protection

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1 Introduction

Dynamic state estimation (DSE) is a generalization of differential protection that offers a reduced likelyhood of misoperation, particularly in the case of devices with nonlinear characteristics such as transformers which are being energized [1]. It is also useful in cases where distance protection performs poorly, such as transmission lines with series compensation [2] or mutually coupled transmission lines [3]. DSE has been previously applied to microgrid branch protection [4, 5, 6].

This study investigates the use of DSE for protection of radial portions of a microgrid or distribution system. This can be a challenge in microgrids or distribution systems with distributed generation on account of lack of fault current from inverter-interfaced generation [7], varying fault current between grid-connected and islanded modes [7], the potential for normally-meshed operation [8] and unbalanced operation due to single-phase loads [8]. Admittance relaying has been investigated as a solution for protection of microgrids [9], though it has been observed to have issues with grounded-wye connected loads [10], and additional relaying is necessary to prevent misoperation [8].

This study treats radial portions of an electrical network as load busses. It is assumed that these portions contain no loops or downstream generation. They are modeled as constant-impedance networks with unknown impedances but known connectivity. To ensure that the number of measured variables is greater than approximately 1.6 times the number of free parameters, most models presented here make the assumption that the loads are balanced, where 1.6 is a commonly selected number to ensure sufficient measurements for system identification [11]. Every load and fault configuration will require a separate model. For a given load configuration, a model for each fault configuration is fit to measured values and the the model with the lowest error in terms of fitting the observed variables is assumed to be the correct one. On a grounded-wye-connected load the following models would be necessary to distinguish between normal operation, line-ground faults and line-line faults:

1. Normal operation: each branch of the load has the same impedance which is modeled as a series resistive-inductive (RL) network

2. Phase A-ground fault: the faulted branch A is modeled as a resistance while the unfaulted branches B and C are modeled as series RL networks with equal parameters

3. Phase B-ground fault: the faulted branch B is modeled as a resistance while the unfaulted branches C and A are modeled as series RL networks with equal parameters

4. Phase C-ground fault: the faulted branch C is modeled as a resistance while the unfaulted branches A and B are modeled as series RL networks with equal parameters

5. Phase A-B fault: the fault impedance is modeled as a resistance across the load terminals A and B, while each branch of the load is modeled as a series RL network

6. Phase B-C fault: the fault impedance is modeled as a resistance across the load terminals B and C, while each branch of the load is modeled as a series RL network

7. Phase C-A fault: the fault impedance is modeled as a resistance across the load terminals C and A, while each branch of the load is modeled as a series RL network.

On a delta-connected system, the following models would be necessary to distinguish between normal operation, line-ground and line-line faults:

1. Normal operation: each branch of the load has the same impedance which is modeled as series RL network

2. Phase A-ground fault: the fault impedance is modeled as a resistance between load terminal A and ground while the load branches are modeled as series RL networks

3. Phase B-ground fault: the fault impedance is modeled as a resistance between load terminal B and ground while the load branches are modeled as series RL networks

4. Phase C-ground fault: the fault impedance is modeled as a resistance between load terminal C and ground while the load branches are modeled as series RL networks

5. Phase A-B fault: the fault impedance is modeled as a resistance across the load terminals A and B, while the the branches across load terminals B-C and C-A are modeled as series RL networks

6. Phase B-C fault: the fault impedance is modeled as a resistance across the load terminals B and C, while the the branches across load terminals C-A and A-B are modeled as series RL networks

7. Phase C-A fault: the fault impedance is modeled as a resistance across the load terminals C and A, while the the branches across load terminals A-B and B-C are modeled as series RL networks.

Both phasor-based and dynamic approaches are investigated for protection. Section 2 describes the implementation of phasor-based state estimation for load bus protection. Phasor-based state estimation is conceptually similar to DSE but more straightforward to derive and implement as it only requires a single time period. Section 3 describes the implementation of DSE for load bus protection. Section 4 describes how two different transient models of loads are developed as test cases and run to test both phasor and dynamic state estimation, while section 5 presents the performance of state estimation on the test cases. Finally, section 6 summarizes conclusions of this study.

2 Phasor Implementation

The phasor implemementation of state estimation-based protection is simpler, so protection will be demonstrated first for the phasor case. For the phasor case, only a single time sample is used which limits the number of measurements and therefore the number of parameters that can be estimated.

2.1 Single-Phase Impedance

The output of the system illustrated in Fig. 1 is

$$\mathbf{y} = \begin{bmatrix} V \\ I \end{bmatrix}. \tag{1}$$

where V and I are phasor quantities. The state of the system is

$$\mathbf{x} = \begin{bmatrix} Z \\ I_z \end{bmatrix}. \tag{2}$$

The output state mapping for the system is the vector-valued function

$$\mathbf{y} = \mathbf{h}(\mathbf{x}) \tag{3}$$

where

$$\begin{aligned} h_1(\mathbf{x}) &= V_z && (4) \\ &= Z I_z && (5) \\ h_2(\mathbf{x}) &= I_z. && (6) \end{aligned}$$

The Jacobian of $\mathbf{h}(\mathbf{x})$ is determined as follows

$$\begin{aligned} \frac{\partial V_z}{\partial Z} &= \frac{\partial}{\partial Z} Z I_z = I_z && (7) \\ \frac{\partial V_z}{\partial I_z} &= \frac{\partial}{\partial I_z} Z I_z = Z && (8) \\ \frac{\partial I_z}{\partial Z} &= \frac{\partial}{\partial Z} I_z = 0 && (9) \\ \frac{\partial I_z}{\partial I_z} &= \frac{\partial}{\partial I_z} I_z = 1. && (10) \end{aligned}$$

The mapping between variables and the state vector is

$$\begin{aligned} Z &= x_1 && (11) \\ I_z &= x_2. && (12) \end{aligned}$$

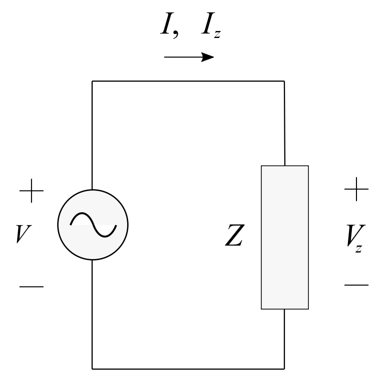


Figure 1: Phasor model for a single-phase load

Given the variable and state mapping, the Jacobian can be built as follows

$$H = \begin{bmatrix} \frac{\partial V_z}{\partial Z} & \frac{\partial V_z}{\partial I_z} \\ \frac{\partial I_z}{\partial Z} & \frac{\partial I_z}{\partial I_z} \end{bmatrix}. \tag{13}$$

Given the Jacobian, the state of the system can be solved for iteratively

$$\boldsymbol{\epsilon}_i = \mathbf{y} - \mathbf{h}(\mathbf{x}_i) \tag{14}$$

$$J_i = ||\boldsymbol{\epsilon}_i||^2 \tag{15}$$

$$\mathbf{x}_{i+1} = \mathbf{x}_i + (H_i' H_i)^{-1} H_i' \boldsymbol{\epsilon}_i. \tag{16}$$

The derivation of eq. 16 is presented in the appendix.

2.2 Grounded-Wye with Line-Ground Fault

The output for the system illustrated in Fig 2 is

$$\mathbf{y} = \begin{bmatrix} I_a & I_b & I_c & V_a & V_b & V_c \end{bmatrix}^T. \tag{17}$$

This is easiest to model as an unbalanced load where the fault impedance is not treated specially. The state of the system is therefore

$$\mathbf{x} = \begin{bmatrix} Y_a & Y_b & Y_c & V_{za} & V_{zb} & V_{zc} \end{bmatrix}^T. \tag{18}$$

The output-state mapping function is

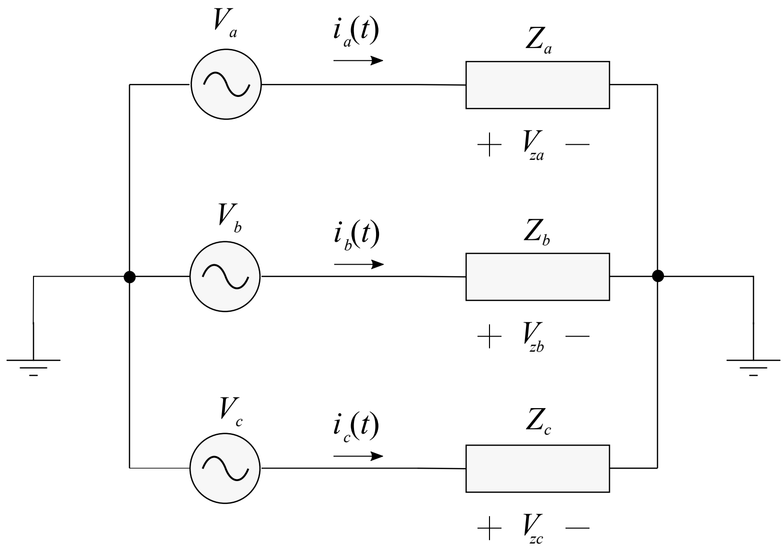


Figure 2: Phasor model for a grounded-wye load with line-ground fault

$$
\begin{aligned}
\mathbf{h}_1(\mathbf{x}) &= I_a = y_a V_a \\
\mathbf{h}_2(\mathbf{x}) &= I_b = y_b V_b \\
\mathbf{h}_3(\mathbf{x}) &= I_c = y_c V_c \\
\mathbf{h}_4(\mathbf{x}) &= V_a = V_{za} \\
\mathbf{h}_5(\mathbf{x}) &= V_b = V_{zb} \\
\mathbf{h}_6(\mathbf{x}) &= V_c = V_{zc}.
\end{aligned} \tag{19}
$$

The Jacobian of $\mathbf{h}(\mathbf{x})$ is determined as follows:

$$
\begin{aligned}
\frac{\partial I_a}{\partial y_a} &= \frac{\partial}{\partial y_a} y_a V_a \\
\frac{\partial I_b}{\partial y_b} &= \frac{\partial}{\partial y_b} y_b V_b \\
\frac{\partial I_c}{\partial y_c} &= \frac{\partial}{\partial y_c} y_a V_c \\
\frac{\partial V_a}{\partial V_{za}} &= \frac{\partial V_a}{\partial V_a} = 1 \\
\frac{\partial V_b}{\partial V_{zb}} &= \frac{\partial V_b}{\partial V_b} = 1 \\
\frac{\partial V_a}{\partial V_{zc}} &= \frac{\partial V_c}{\partial V_c} = 1.
\end{aligned} \tag{20}
$$

The mapping between variables and the state vector is

$$\begin{aligned}
y_a &= x_1 \\
y_b &= x_2 \\
y_c &= x_3 \\
V_{za} &= x_4 \\
V_{zb} &= x_5 \\
V_{zc} &= x_6.
\end{aligned} \tag{21}$$

Given the variable and state vector mapping, the Jacobian can be built as follows

$$\begin{aligned}
H(1,1) &= \frac{\partial I_a}{\partial y_a} \\
H(2,2) &= \frac{\partial I_b}{\partial y_b} \\
H(3,3) &= \frac{\partial I_c}{\partial y_c} \\
H(4,4) &= \frac{\partial V_a}{\partial V_{za}} = 1 \\
H(5,5) &= \frac{\partial V_b}{\partial V_{zb}} = 1 \\
H(6,6) &= \frac{\partial V_c}{\partial V_{zc}} = 1.
\end{aligned} \tag{22}$$

2.3 Grounded-Wye with Line-Line Fault

The output for the system illustrated in Fig 3 is

$$\mathbf{y} = \begin{bmatrix} I_a & I_b & I_c & V_a & V_b & V_c \end{bmatrix}^T. \tag{23}$$

The state of the system is

$$\mathbf{x} = \begin{bmatrix} Y_l & Y_f & V_{za} & V_{zb} & V_{zc} \end{bmatrix}^T. \tag{24}$$

The output state mapping for the system is the vector-valued function

$$\mathbf{y} = \mathbf{h}(\mathbf{x}). \tag{25}$$

This is derived from the admittance matrix of the system

$$Y = \begin{bmatrix} y_l + y_f & -y_f & 0 \\ -y_f & y_l & 0 \\ 0 & 0 & y_l \end{bmatrix}. \tag{26}$$

From the relation

$$I = YV \tag{27}$$

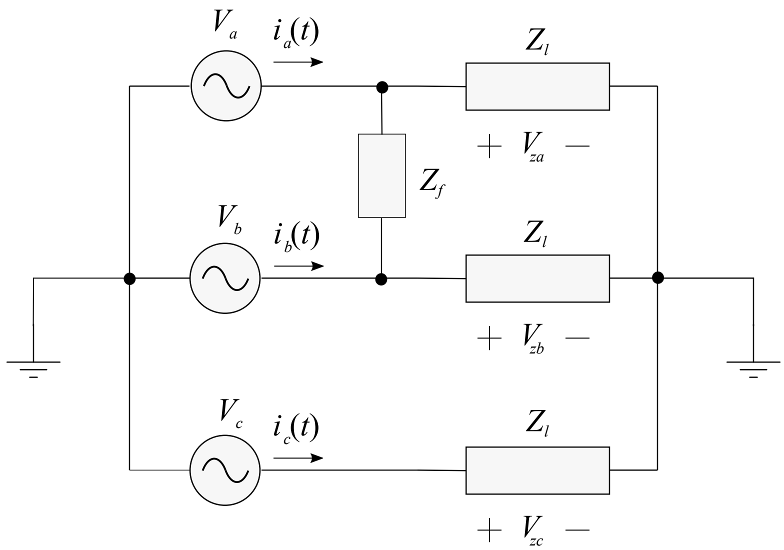


Figure 3: Phasor model for a grounded-wye load with line-line fault

$\mathbf{h}(\mathbf{x})$ can be derived

$$
\begin{aligned}
h_1(\mathbf{x}) &= I_a = (y_l + y_f)V_{za} - y_f V_{zb} \\
h_2(\mathbf{x}) &= I_b = -y_f V_{za} + y_l V_{zb} \\
h_3(\mathbf{x}) &= I_c = y_l V_{zc} \\
h_4(\mathbf{x}) &= V_a = V_{za} \\
h_5(\mathbf{x}) &= V_b = V_{zb} \\
h_6(\mathbf{x}) &= V_c = V_{zc}.
\end{aligned} \tag{28}
$$

The Jacobian of $\mathbf{h}(\mathbf{x})$ is determined as follows

$$
\begin{aligned}
\frac{\partial I_a}{\partial y_l} &= \frac{\partial}{\partial y_l}((y_l + y_f)V_{za} - y_f V_{zb}) \\
&= \frac{\partial}{\partial y_l}(V_{za}y_l + (V_{za} - V_{zb})y_f) \\
&= V_{za} \\
\frac{\partial I_a}{\partial y_f} &= \frac{\partial}{\partial y_f}((y_l + y_f)V_{za} - ty_f V_{zb}) \\
&= \frac{\partial}{\partial y_f}((V_{za} - V_{zb})y_f + V_{za}y_l) \\
&= V_{za} - V_{zb} \\
\frac{\partial I_b}{\partial y_l} &= \frac{\partial}{\partial y_l}(-y_f V_{za} + y_l V_{zb}) \\
&= V_{zb} \\
\frac{\partial I_b}{\partial y_f} &= \frac{\partial}{\partial y_f}(-y_f V_{za} + y_l V_{zb}) \\
&= -V_{za} \\
\frac{\partial I_c}{\partial y_l} &= \frac{\partial}{\partial y_l} V_z y_l \\
&= V_{zc} \\
\frac{\partial I_c}{\partial y_f} &= \frac{\partial}{\partial y_f} V_z y_l \\
&= 0 \qquad (29) \\
\frac{\partial V_a}{\partial V_{za}} &= \frac{\partial V_a}{\partial V_a} = 1 \\
\frac{\partial V_b}{\partial V_{zb}} &= \frac{\partial V_b}{\partial V_b} = 1 \\
\frac{\partial V_a}{\partial V_{zc}} &= \frac{\partial V_c}{\partial V_c} = 1. \qquad (30)
\end{aligned}
$$

2.4 Delta-Connected Load with Line-Line Fault

The output of the system illustrated in Fig. 4 is

$$\mathbf{y} = \begin{bmatrix} I_a & I_b & I_c & V_a & V_b & V_c \end{bmatrix}^T \qquad (31)$$

where V and I are phasor quantities.
The state of the system is

$$\mathbf{x} = \begin{bmatrix} Y_f & Y_{ll} & V_{za} & V_{zb} & V_{zc} \end{bmatrix}^T. \qquad (32)$$

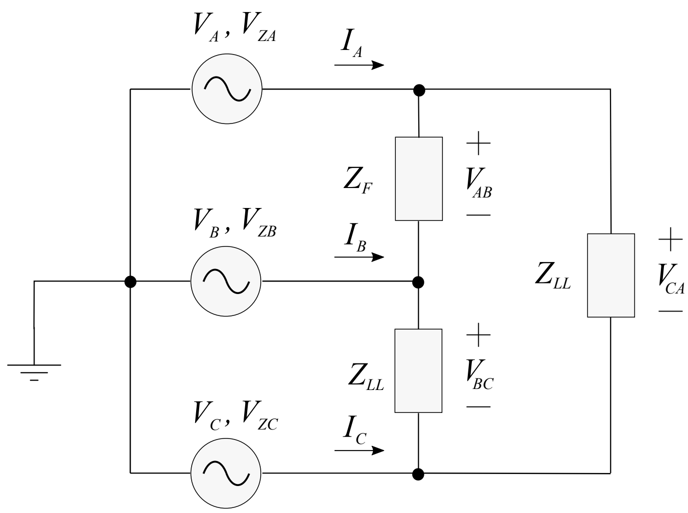


Figure 4: Phasor model for a delta-connected load with a line-line fault

The output state mapping for the system is the vector-valued function

$$\mathbf{y} = \mathbf{h}(\mathbf{x}). \tag{33}$$

This is derived from the admittance matrix of the system

$$Y = \begin{bmatrix} y_{aa} & -y_{ab} & -y_{ac} \\ -y_{ab} & y_{bb} & -y_{bc} \\ -y_{ac} & -y_{bc} & y_{cc} \end{bmatrix} \tag{34}$$

where

$$y_{aa} = y_{ab} + y_{ca} \tag{35}$$

$$y_{bb} = y_{ab} + y_{bc} \tag{36}$$

$$y_{cc} = y_{ca} + y_{bc} \tag{37}$$

and

$$y_{ab} = y_f \tag{38}$$

$$y_{bc} = y_{ca} = y_{ll}. \tag{39}$$

From the relation

$$I = YV \tag{40}$$

$\mathbf{h}(\mathbf{x})$ can be derived

$$
\begin{aligned}
h_1(\mathbf{x}) &= I_a = y_{aa}V_{za} - y_{ab}V_{zb} - y_{ca}V_{zc} \\
&= (y_{ab} + y_{ca})V_{za} - y_{ab}V_{zb} - y_{ca}V_{zc} \\
&= (y_f + y_{ll})V_{za} - y_f V_{sb} - y_{ll}V_{zc} \\
h_2(\mathbf{x}) &= I_b = -y_{ab}V_{za} + y_{bb}V_{zb} - y_{bc}V_{zc} \\
&= -y_{ab}V_{za} + (y_{ab} + y_{bc})V_{zb} - y_{bc}V_{zc} \\
&= -y_f V_{za} + (y_f + y_{ll})V_{zb} - y_{ll}V_{zc} \\
h_3(\mathbf{x}) &= I_c = -y_{ca}V_{za} - y_{bc}V_{zb} + y_{cc}V_{zc} \\
&= -y_{ca}V_{za} - y_{bc}V_{zb} + (y_{ac} + y_{bc})V_{zc} \\
&= -y_{ll}V_{za} - y_{ll}V_{zb} + 2y_{ll}V_{zc} \\
h_4(\mathbf{x}) &= V_a = V_{za} \\
h_5(\mathbf{x}) &= V_b = V_{zb} \\
h_6(\mathbf{x}) &= V_c = V_{zc}.
\end{aligned}
\tag{41}
$$

The Jacobian of $\mathbf{h}(\mathbf{x})$ is determined as follows

$$
\begin{aligned}
\frac{\partial I_a}{\partial y_f} &= \frac{\partial}{\partial y_f}((y_f + y_{ll})V_{za} - y_f V_{sb} - y_{ll} V_{zc}) \\
&= \frac{\partial}{\partial y_f}(V_{za} - V_{zb})y_f + (V_{za} - V_{zc})y_{ll} \\
&= V_{za} - V_{zb} \\
\frac{\partial I_a}{\partial y_l l} &= \frac{\partial}{\partial y_{ll}}((y_f + y_{ll})V_{za} - y_f V_{sb} - y_{ll} V_{zc}) \\
&= \frac{\partial}{\partial y_{ll}}(V_{za} - V_{zb})y_f + (V_{za} - V_{zc})y_{ll} \\
&= V_{za} - V_{zc} \\
\frac{\partial I_b}{\partial y_f} &= \frac{\partial}{\partial y_f}(-y_f V_{za} + (y_f + y_{ll})V_{zb} - y_{ll} V_{zc}) \\
&= \frac{\partial}{\partial y_f}((V_{zb} - V_{za})y_f + (V_{zb} - V_{zc})y_{ll}) \\
&= V_{zb} - V_{za} \\
\frac{\partial I_b}{\partial y_{ll}} &= \frac{\partial}{\partial y_{ll}}(-y_f V_{za} + (y_f + y_{ll})V_{zb} - y_{ll} V_{zc}) \\
&= \frac{\partial}{\partial y_f}((V_{zb} - V_{za})y_f + (V_{zb} - V_{zc})y_{ll}) \\
&= V_{zb} - V_{zc} \\
\frac{\partial I_c}{\partial y_f} &= \frac{\partial}{\partial y_f}(-y_{ll} V_{za} - y_{ll} V_{zb} + 2y_{ll} V_{zc}) \\
&= \frac{\partial}{\partial y_f}((2V_{zc} - V_{za} - V_{zb})y_{ll}) \\
&= 0 \\
\frac{\partial I_c}{\partial y_{ll}} &= \frac{\partial}{\partial y_{ll}}(-y_{ll} V_{za} - y_{ll} V_{zb} + 2y_{ll} V_{zc}) \\
&= \frac{\partial}{\partial y_{ll}}((2V_{zc} - V_{za} - V_{zb})y_{ll}) \\
&= 2V_{zc} - V_{za} - V_{zb} \\
\frac{\partial V_a}{\partial V_{za}} &= \frac{\partial V_a}{\partial V_a} = 1 \\
\frac{\partial V_b}{\partial V_{zb}} &= \frac{\partial V_b}{\partial V_b} = 1 \\
\frac{\partial V_a}{\partial V_{zc}} &= \frac{\partial V_c}{\partial V_c} = 1.
\end{aligned} \tag{42}
$$

The mapping between variables and the state vector is

$$\begin{aligned} y_f &= x_1 \\ y_{ll} &= x_2 \\ V_{za} &= x_3 \\ V_{zb} &= x_4 \\ V_{zc} &= x_5. \end{aligned} \tag{43}$$

Given the variable and state vector mapping, the Jacobian can be built as follows

$$\begin{aligned} H(1,1) &= \frac{\partial I_a}{\partial y_f} & H(1,2) &= \frac{\partial I_a}{\partial y_{ll}} \\ H(2,1) &= \frac{\partial I_b}{\partial y_f} & H(2,2) &= \frac{\partial I_b}{\partial y_{ll}} \\ H(3,1) &= \frac{\partial I_c}{\partial y_f} & H(3,2) &= \frac{\partial I_c}{\partial y_{ll}} \\ H(4,3) &= \frac{\partial V_a}{\partial V_{za}} = 1 \\ H(5,4) &= \frac{\partial V_b}{\partial V_{zb}} = 1 \\ H(5,4) &= \frac{\partial V_c}{\partial V_{zc}} = 1. \end{aligned} \tag{44}$$

Unless otherwise specified $H(n,m) = 0$.

2.5 Delta-Connected Load with a Line-Ground Fault

The output of the system illustrated in Fig. 5 is

$$\mathbf{y} = \begin{bmatrix} I_a & I_b & I_c & V_a & V_b & V_c \end{bmatrix}^T. \tag{45}$$

where V_ϕ and I_ϕ are phasor quantities.
The state of the system is

$$\mathbf{x} = \begin{bmatrix} Y_{ll} & Y_f & V_{za} & V_{zb} & V_{zc} \end{bmatrix}^T. \tag{46}$$

In the above, the output is derived from the admittance matrix described in Eqs. 26 with the distinction that

$$y_{aa} = y_{ab} + y_{ca} + y_{ag} \tag{47}$$

$$y_{bb} = y_{ab} + y_{bc} \tag{48}$$

$$y_{cc} = y_{ca} + y_{bc} \tag{49}$$

and

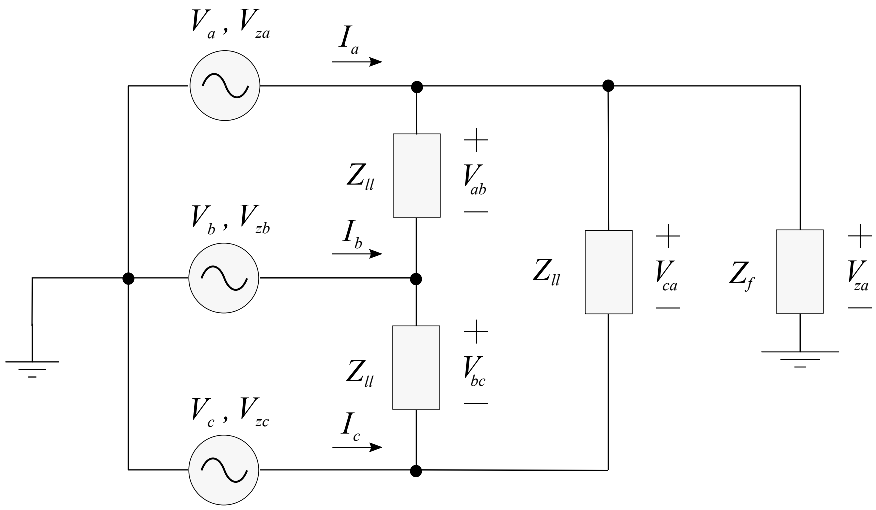


Figure 5: Phasor model for a delta-connected load with a line-ground fault

$$y_{ag} = y_f \tag{50}$$

$$y_{ab} = y_{bc} = y_{ca} = y_{ll}. \tag{51}$$

The output-state mapping function is derived similarly as in Section 2.4:

$$\begin{aligned}
h_1(\mathbf{x}) &= I_a = y_{aa}V_{za} - y_{ab}V_{zb} - y_{ca}V_{zc} \\
&= (y_{ab} + y_{ca} + y_{ag})V_{za} - y_{ab}V_{zb} - y_{ca}V_{zc} \\
&= (y_f + 2y_{ll})V_{za} - y_{ll}V_{sb} - y_{ll}V_{zc} \\
h_2(\mathbf{x}) &= I_b = -y_{ab}V_{za} + y_{bb}V_{zb} - y_{bc}V_{zc} \\
&= -y_{ab}V_{za} + (y_{ab} + y_{bc})V_{zb} - y_{bc}V_{zc} \\
&= -y_{ll}V_{za} + 2y_{ll}V_{zb} - y_{ll}V_{zc} \\
h_3(\mathbf{x}) &= I_c = -y_{ca}V_{za} - y_{bc}V_{zb} + y_{cc}V_{zc} \\
&= -y_{ca}V_{za} - y_{bc}V_{zb} + (y_{ac} + y_{bc})V_{zc} \\
&= -y_{ll}V_{za} - y_{ll}V_{zb} + 2y_{ll}V_{zc} \\
h_4(\mathbf{x}) &= V_a = V_{za} \\
h_5(\mathbf{x}) &= V_b = V_{zb} \\
h_6(\mathbf{x}) &= V_c = V_{zc}.
\end{aligned} \tag{52}$$

The Jacobian of $\mathbf{h}(\mathbf{x})$ is determined as follows

$$
\begin{aligned}
\frac{\partial I_a}{\partial y_{ll}} &= \frac{\partial}{\partial y_{ll}}((y_f + 2y_{ll})V_{za} - y_{ll}V_{sb} - y_{ll}V_{zc}) \\
&= \frac{\partial}{\partial y_{ll}}((2V_{za} - V_{zb} - V_{zc})y_{ll} + V_{za}y_f) \\
&= 2V_{za} - V_{zb} - V_{zc} \\
\frac{\partial I_a}{\partial y_f} &= \frac{\partial}{\partial y_f}((y_f + 2y_{ll})V_{za} - y_{ll}V_{sb} - y_{ll}V_{zc}) \\
&= \frac{\partial}{\partial y_f}((2V_{za} - V_{zb} - V_{zc})y_{ll} + V_{za}y_f) \\
&= V_{za} \\
\frac{\partial I_b}{\partial y_{ll}} &= \frac{\partial}{\partial y_{ll}}(-y_{ll}V_{za} + 2y_{ll}V_{zb} - y_{ll}V_{zc}) \\
&= \frac{\partial}{\partial y_{ll}}(-V_{za}y_{ll} + 2V_{zb} - V_{zc})y_{ll}) \\
&= 2V_{zb} - V_{za} - V_{zc} \\
\frac{\partial I_b}{\partial y_f} &= \frac{\partial}{\partial y_f}(-y_{ll}V_{za} + 2y_{ll}V_{zb} - y_{ll}V_{zc}) \\
&= 0 \\
\frac{\partial I_c}{\partial y_{ll}} &= \frac{\partial}{\partial y_{ll}}(-y_{ll}V_{za} - y_{ll}V_{zb} + 2y_{ll}V_{zc}) \\
&= \frac{\partial}{\partial y_{ll}}(-V_{za} + 2V_{zc} - V_{zb})y_{ll}) \\
&= 2V_{zc} - V_{za} - V_{zb} \\
\frac{\partial I_c}{\partial y_f} &= \frac{\partial}{\partial y_f}(-y_{ll}V_{za} - y_{ll}V_{zb} + 2y_{ll}V_{zc}) \\
&= 0 \\
\frac{\partial V_a}{\partial V_{za}} &= \frac{\partial V_a}{\partial V_a} = 1 \\
\frac{\partial V_b}{\partial V_{zb}} &= \frac{\partial V_b}{\partial V_b} = 1 \\
\frac{\partial V_a}{\partial V_{zc}} &= \frac{\partial V_c}{\partial V_c} = 1.
\end{aligned} \tag{53}
$$

The mapping between variables and the state vector is

$$\begin{aligned}
y_{ll} &= x_1 \\
y_f &= x_2 \\
V_{za} &= x_3 \\
V_{zb} &= x_4 \\
V_{zc} &= x_5.
\end{aligned} \tag{54}$$

Given the variable and state vector mapping, the Jacobian can be built as follows

$$\begin{aligned}
H(1,1) &= \frac{\partial I_a}{\partial y_{ll}} & H(1,2) &= \frac{\partial I_a}{\partial y_f} \\
H(2,1) &= \frac{\partial I_b}{\partial y_{ll}} & H(2,2) &= \frac{\partial I_b}{\partial y_f} \\
H(3,1) &= \frac{\partial I_c}{\partial y_{ll}} & H(3,2) &= \frac{\partial I_c}{\partial y_f} \\
H(4,3) &= \frac{\partial V_a}{\partial V_{za}} = 1 \\
H(5,4) &= \frac{\partial V_b}{\partial V_{zb}} = 1 \\
H(5,4) &= \frac{\partial V_c}{\partial V_{zc}} = 1.
\end{aligned} \tag{55}$$

3 Dynamic Implementation

As in the case of the previous section, the dynamic implementation of DSE is applied to single-phase, grounded-wye and delta-connected load configurations. While the phasor implementation uses a single time point for state estimation, with the dynamic implementation several points are used, in this case 12 cycles sampled at a 2 kHz sample rate.

3.1 Single-Phase Series RL Load

The output for the system illustrated in Fig. 6 is

$$y(t) = \begin{bmatrix} v(t) \\ i(t)z(t) \end{bmatrix}. \tag{56}$$

For the purposes of state estimation, this is sampled at points $n \in \{1, ..., N\}$ giving the vector-value equation

$$\mathbf{y} = \begin{bmatrix} \mathbf{v} \\ \mathbf{i} \end{bmatrix}. \tag{57}$$

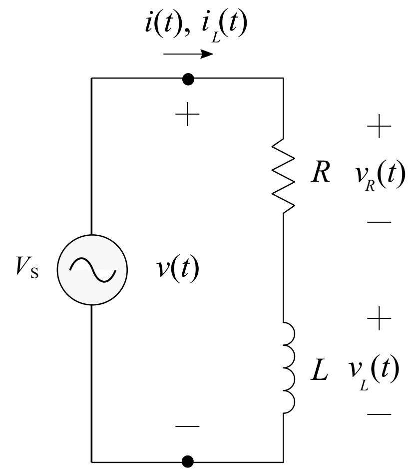


Figure 6: Dynamic model for single-phase RL series load

where $\mathbf{v} = \begin{bmatrix} v(1) & v(2) & \cdots & v(N) \end{bmatrix}^T$, $\mathbf{i} = \begin{bmatrix} i(1) & i(2) & \cdots & i(N) \end{bmatrix}^T$ and $\mathbf{z} = \begin{bmatrix} z(1) & z(2) & \cdots & z(N) \end{bmatrix}^T$.
The state for the system is

$$\mathbf{x} = \begin{bmatrix} R & L & \mathbf{v}_r & \mathbf{v}_l \end{bmatrix}^T. \tag{58}$$

The output-state mapping for the system is the vector-valued function

$$\mathbf{y} = \mathbf{h}(\mathbf{x}) \tag{59}$$

where

$$\begin{aligned} h_n(\mathbf{x}) &= v_r(n) + v_l(n) \quad \forall n \in \{1, 2, \ldots, N\} \\ h_{N+n}(\mathbf{x}) &= Gv_r(n) \quad \forall n \in \{1, 2, \ldots, N\} \\ h_{2N+n}(\mathbf{x}) &= Gv_r(n) - Gv_r(n-2) + \frac{2\Lambda\Delta t}{6}(v_l(n) + 4v_l(n-1) + v_l(n-2)), \quad \forall n \in \{3, 4, \ldots, N\}. \end{aligned} \tag{60}$$

In the above $v_R(n) = Ri_L(n)$ follows from discretizing $v_R(t) = Ri_L(t)$ and

$$v_l(n) = \frac{2\Lambda\Delta t}{6}(v_l(n) + 4v_l(n-1) + v_l(n-2))$$

follows from discretizing

$$i_l(t) = \frac{1}{L}\int_{t-\Delta t}^{t} v_l(\tau) d\tau$$

via Simpson's 1/3 rule [12].

Given the variable and state vector mapping, the Jacobian can be built as follows
The Jacobian of $\mathbf{h}(\mathbf{x})$ is determined as follows

$$
\begin{aligned}
H(n,2+n) &= \frac{\partial v(n)}{\partial v_r(n)} = 1 \quad \forall n \in \{1,2,\dots,N\} \\
H(n,2+N+n) &= \frac{\partial v(n)}{\partial v_l(n)} = 1 \quad \forall n \in \{1,2,\dots,N\} \\
H(N+n,1) &= \frac{\partial i(n)}{\partial G} = v_r(n) \quad \forall n \in \{1,2,\dots,N\} \\
H(N+n,2+n) &= \frac{\partial i(n)}{\partial v_r(n)} = G \quad \forall n \in \{1,2,\dots,N\} \\
H(2N+n-2,1) &= \frac{\partial z(n-2)}{\partial R} = v_r(n) - v_r(n-2) \quad \forall n \in \{3,4,\dots,N\} \\
H(2N+n-2,n) &= \frac{\partial z(n-2)}{\partial v_r(n)} = G \quad \forall n \in \{3,4,\dots,N\} \\
H(2N+n-2,n) &= \frac{\partial z(n-2)}{\partial v_r(n-2)} = -G \quad \forall n \in \{3,4,\dots,N\} \\
H(2N+n-2,2) &= \frac{\partial z(n-2)}{\partial \Lambda} = \frac{2\Delta t}{6}(v_l(n) + 4v_l(n-1) + v_l(n-2)) \\
&\forall n \in \{3,4,\dots,N\} \\
H(2N+n-2,2+N+n) &= \frac{\partial z(n-2)}{\partial v_l(n)} = \frac{\Delta t \Lambda}{3} \quad \forall n \in \{1,2,\dots,N\} \\
H(2N+n-2,1+N+n) &= \frac{\partial z(n-2)}{\partial v_l(n-1)} = \frac{4\Delta t \Lambda}{3} \quad \forall n \in \{1,2,\dots,N\} \\
H(2N+n-2,N+n) &= \frac{\partial z(n-2)}{\partial v_l(n-2)} = \frac{\Delta t \Lambda}{3} \quad \forall n \in \{1,2,\dots,N\} \qquad (61)
\end{aligned}
$$

In the above, $H(n,m) = 0$ unless otherwise specified. The state of the system can then be solved for by applying eqs 14–16.

3.2 Grounded-Wye Load without Fault

The sampled output of the system illustrated in Fig. 7 is

$$\mathbf{y} = \begin{bmatrix} \mathbf{v}_a & \mathbf{v}_b & \mathbf{v}_c & \mathbf{i}_a & \mathbf{i}_b & \mathbf{i}_c & \mathbf{z}_a & \mathbf{z}_b & \mathbf{z}_c \end{bmatrix}^T \qquad (62)$$

where $\mathbf{v}_\phi = \begin{bmatrix} v_\phi(1) & v_\phi(2) & \cdots & v_\phi(N) \end{bmatrix}^T$, $\mathbf{i}_\phi = \begin{bmatrix} i_\phi(1) & i_\phi(2) & \cdots & i_\phi(N) \end{bmatrix}^T$ and $\mathbf{z}_\phi = \begin{bmatrix} z_\phi(1) & z_\phi(2) & \cdots & z_\phi(N-2) \end{bmatrix}^T$ for $\phi \in \{a,b,c\}$. The state for the system is

$$x(t) = \begin{bmatrix} G & \Lambda & \mathbf{v}_{ra} & \mathbf{v}_{rb} & \mathbf{v}_{rc} & \mathbf{v}_{la} & \mathbf{v}_{lb} & \mathbf{v}_{lc} \end{bmatrix}^T . \qquad (63)$$

where $G = R^{-1}$ is the conductance, $\Lambda = L^{-1}$ is the reciprocal of the inductance, $\mathbf{v}_{r\phi}$ is the voltage across the resistance on phase ϕ at each time period $1,\dots,N$

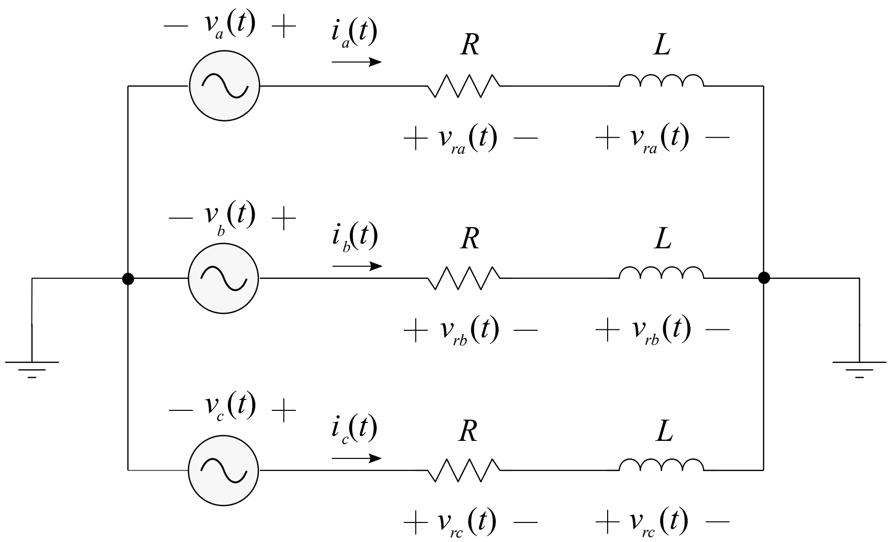


Figure 7: Dynamic model for a grounded-wye-connected RL load

and $\mathbf{v}_{l\phi}$ is the voltage across the inductance on phase ϕ at each time period $1, \ldots, N$. The output state-mapping for the system is given by

$$v_\phi(n) = v_{r\phi}(n) + v_{l\phi}(n) \quad \forall \phi \in \{a, b, c\}, n \in \{1, 2, \ldots, N\}$$
$$i_\phi(n) = G v_{r\phi}(n) \quad \forall \phi \in \{a, b, c\}, n \in \{1, 2, \ldots, N\}$$

$$z_\phi(n-2) = G(v_{r\phi}(n) - v_{r\phi}(n-2)) - \frac{2\Delta t \Lambda}{6}(v_{l\phi}(n) + 4v_{l\phi}(n-1) + v_{l\phi}(n-2))$$
$$\forall \phi \in \{a, b, c\}, n \in \{3, 4, \ldots, N\}. \tag{64}$$

In Eq. 64 $z_\phi(n)$ follows from discretizing

$$i_\phi(t) = \Lambda \int_{t-\Delta t}^{t} v_{l\phi}(\tau) d\tau \tag{65}$$

via Simpson's 1/3 rule [12]. The output function $\mathbf{h}(\mathbf{x})$ can be written as

$$
\begin{aligned}
h_n(\mathbf{x}) &= v_{ra}(n) + v_{la}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+N}(\mathbf{x}) &= v_{rb}(n) + v_{lb}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+2N}(\mathbf{x}) &= v_{rb}(n) + v_{lb}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+3N}(\mathbf{x}) &= Gv_{ra}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+4N}(\mathbf{x}) &= Gv_{rb}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+5N}(\mathbf{x}) &= Gv_{rc}(n)) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+6N}(\mathbf{x}) &= G(v_{ra}(n) - v_{ra}(n-2)) - \frac{2\Delta t \Lambda}{6}(v_{la}(n) + 4v_{la}(n-1) + v_{la}(n-2)) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+7N}(\mathbf{x}) &= G(v_{rb}(n) - v_{rb}(n-2)) - \frac{2\Delta t \Lambda}{6}(v_{lb}(n) + 4v_{lc}(n-1) + v_{lb}(n-2)) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+8N}(\mathbf{x}) &= G(v_{rc}(n) - v_{rc}(n-2)) - \frac{2\Delta t \Lambda}{6}(v_{lc}(n) + 4v_{lc}(n-1) + v_{lc}(n-2)) \quad \forall n \in \{1, 2, \ldots, N\}.
\end{aligned} \tag{66}
$$

Given the variable and state vector mapping, the Jacobian can be built as follows

$$
\begin{aligned}
H(n, 2+n) &= \frac{\partial v_a(n)}{\partial v_{ra}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\} \\
H(N+n, 2+N+n) &= \frac{\partial v_b(n)}{\partial v_{rb}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\} \\
H(2N+n, 2+2N+n) &= \frac{\partial v_c(n)}{\partial v_{rc}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\} \\
H(n, 2+3N+n) &= \frac{\partial v_a(n)}{\partial v_{la}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\} \\
H(N+n, 2+4N+n) &= \frac{\partial v_b(n)}{\partial v_{lb}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\} \\
H(2N+n, 2+5N+n) &= \frac{\partial v_c(n)}{\partial v_{lc}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\} \\
H(3N+n, 1) &= \frac{\partial i_a(n)}{\partial G} = v_{ra}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
H(4N+n, 1) &= \frac{\partial i_b(n)}{\partial G} = v_{rb}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
H(5N+n, 1) &= \frac{\partial i_c(n)}{\partial G} = v_{rc}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
H(3N+n, 2+n) &= \frac{\partial i_a(n)}{\partial v_{ra}(n)} = G \quad \forall n \in \{1, 2, \ldots, N\} \\
H(4N+n, 2+N+n) &= \frac{\partial i_b(n)}{\partial v_{rb}(n)} = G \quad \forall n \in \{1, 2, \ldots, N\} \\
H(5N+n, 2+2N+n) &= \frac{\partial i_c(n)}{\partial v_{rc}(n)} = G \quad \forall n \in \{1, 2, \ldots, N\} \\
H(6N+n-2, 1) &= \frac{\partial z_a(n-2)}{\partial G} = v_{ra}(n) - v_{ra}(n-2) \quad \forall n \in \{3, 4, \ldots, N\} \\
H(7N+n-2, 1) &= \frac{\partial z_b(n-2)}{\partial G} = v_{rb}(n) - v_{rb}(n-2) \quad \forall n \in \{3, 4, \ldots, N\} \\
H(8N+n-2, 1) &= \frac{\partial z_c(n-2)}{\partial G} = v_{rc}(n) - v_{rc}(n-2) \quad \forall n \in \{3, 4, \ldots, N\} \\
H(6N+n-2, 2+n) &= \frac{\partial z_a(n-2)}{\partial v_{ra}(n)} = G \quad \forall n \in \{3, 4, \ldots, N\} \\
H(7N+n-2, 2+N+n) &= \frac{\partial z_b(n-2)}{\partial v_{rb}(n)} = G \quad \forall n \in \{3, 4, \ldots, N\} \\
H(8N+n-2, 2+2N+n) &= \frac{\partial z_c(n-2)}{\partial v_{rc}(n)} = G \quad \forall n \in \{3, 4, \ldots, N\}
\end{aligned}
$$

$$
\begin{aligned}
H(6N+n-2,n) &= \frac{\partial z_a(n-2)}{\partial v_{ra}(n-2)} = -G \quad \forall n \in \{3,4,\ldots,N\} \\
H(7N+n-2,N+n) &= \frac{\partial z_b(n-2)}{\partial v_{rb}(n-2)} = -G \quad \forall n \in \{3,4,\ldots,N\} \\
H(8N+n-2,2N+n) &= \frac{\partial z_c(n-2)}{\partial v_{rc}(n-2)} = -G \quad \forall n \in \{3,4,\ldots,N\} \\
H(6N+n-2,2) &= \frac{\partial z_a(n-2)}{\partial \Lambda} = -\frac{\Delta t}{3}(v_{la}(n)+4v_{la}(n-1)+v_{la}(n-2)) \\
&\forall n \in \{3,4\ldots,N\} \\
H(7N+n-2,2) &= \frac{\partial z_b(n-2)}{\partial \Lambda} = -\frac{\Delta t}{3}(v_{lb}(n)+4v_{lb}(n-1)+v_{lb}(n-2)) \\
&\forall n \in \{3,4\ldots,N\} \\
H(8N+n-2,2) &= \frac{\partial z_c(n-2)}{\partial \Lambda} = -\frac{\Delta t}{3}(v_{lc}(n)+4v_{lc}(n-1)+v_{lc}(n-2)) \\
&\forall n \in \{3,4\ldots,N\} \\
H(6N+n-2,2+3N+n) &= \frac{\partial z_a(n-2)}{\partial v_{la}(n)} = -\frac{\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(7N+n-2,2+4N+n) &= \frac{\partial z_b(n-2)}{\partial v_{lb}(n)} = -\frac{\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(8N+n-2,2+5N+n) &= \frac{\partial z_c(n-2)}{\partial v_{lc}(n)} = -\frac{\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(6N+n-2,1+3N+n) &= \frac{\partial z_a(n-2)}{\partial v_{la}(n-1)} = -\frac{4\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(6N+n-2,1+4N+n) &= \frac{\partial z_b(n-2)}{\partial v_{lb}(n-1)} = -\frac{4\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(6N+n-2,1+5N+n) &= \frac{\partial z_c(n-2)}{\partial v_{lc}(n-1)} = -\frac{4\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(6N+n-2,3N+n) &= \frac{\partial z_a(n-2)}{\partial v_{la}(n-2)} = -\frac{\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(7N+n-2,4N+n) &= \frac{\partial z_b(n-2)}{\partial v_{lb}(n-2)} = -\frac{\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(8N+n-2,5N+n) &= \frac{\partial z_c(n-2)}{\partial v_{lc}(n-2)} = -\frac{\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\}. \qquad (67)
\end{aligned}
$$

The state of the system can then be solved for by applying eqs 14–16.

3.3 Grounded-Wye Load with Line-Ground Fault

The sampled output of the system illustrated in Fig. 8 is

$$
\mathbf{y} = \begin{bmatrix}\mathbf{v}_a & \mathbf{v}_b & \mathbf{v}_c & \mathbf{i}_a & \mathbf{i}_b & \mathbf{i}_c, & \mathbf{z}_b & \mathbf{z}_c\end{bmatrix}^T. \qquad (68)
$$

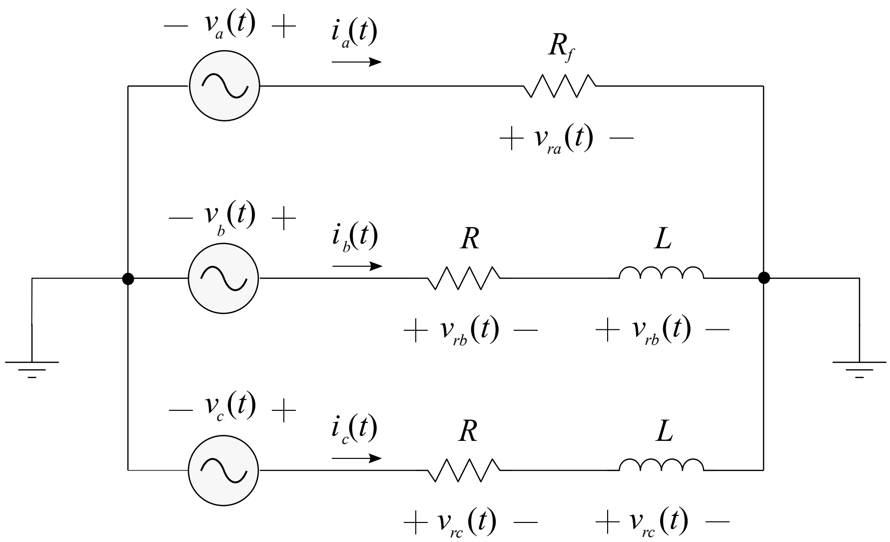


Figure 8: Dynamic model for a grounded-wye-connected RL load with a line-ground fault

Note that there are no $z_a(n)$ output variables as the reactive impedance on phase A is large compared to the parallel fault conductance G_f. The state for the system is

$$x(t) = \begin{bmatrix} G & \Lambda & G_f & \mathbf{v}_{ra} & \mathbf{v}_{rb} & \mathbf{v}_{rc} & \mathbf{v}_{lb} & \mathbf{v}_{lc} \end{bmatrix}^T \tag{69}$$

where $G_f = R^{-1}$ is the conductance and the remaining states are the same as those in that of the grounded-wye no-fault state in eq. 63. The output state-mapping for the system is given by

$$v_\phi(n) = \begin{cases} V_{r\phi}(n) \quad \forall n \in \{1, 2, \ldots, N\} & \phi = a, \\ v_{r\phi}(n) + v_{l\phi}(n) \quad \forall n \in \{1, 2, \ldots, N\} & \phi \in \{b, c\}, \end{cases}$$

$$i_\phi(n) = G v_{r\phi}(n) \quad \forall \phi \in \{a, b, c\}, n \in \{1, 2, \ldots, N\}$$

$$\begin{aligned} z_\phi(n-2) = G(v_{r\phi}(n) - v_{r\phi}(n-2)) - \frac{2\Delta t \Lambda}{6}(v_{l\phi}(n) + 4v_{l\phi}(n-1) + v_{l\phi}(n-2)) \\ \forall \phi \in \{b, c\}, n \in \{3, 4 \ldots, N\}. \end{aligned} \tag{70}$$

The output function $\mathbf{h}(\mathbf{x})$ can be written as

$$
\begin{aligned}
h_n(\mathbf{x}) &= v_{ra}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+N}(\mathbf{x}) &= v_{rb}(n) + v_{lb}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+2N}(\mathbf{x}) &= v_{rb}(n) + v_{lb}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+3N}(\mathbf{x}) &= Gv_{ra}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+4N}(\mathbf{x}) &= Gv_{rb}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+5N}(\mathbf{x}) &= Gv_{rc}(n)) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+6N}(\mathbf{x}) &= G(v_{rb}(n) - v_{rb}(n-2)) - \frac{2\Delta t \Lambda}{6}(v_{lb}(n) + 4v_{lc}(n-1) + v_{lb}(n-2)) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+7N}(\mathbf{x}) &= G(v_{rc}(n) - v_{rc}(n-2)) - \frac{2\Delta t \Lambda}{6}(v_{lc}(n) + 4v_{lc}(n-1) + v_{lc}(n-2)) \quad \forall n \in \{1, 2, \ldots, N\}.
\end{aligned} \tag{71}
$$

Given the variable and state vector mapping, the Jacobian can be built as follows

$$H(n, 3+n) = \frac{\partial v_a(n)}{\partial v_{ra}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\}$$

$$H(N+n, 3+N+n) = \frac{\partial v_b(n)}{\partial v_{rb}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\}$$

$$H(2N+n, 3+2N+n) = \frac{\partial v_c(n)}{\partial v_{rc}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\}$$

$$H(N+n, 3+3N+n) = \frac{\partial v_b(n)}{\partial v_{lb}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\}$$

$$H(2N+n, 3+4N+n) = \frac{\partial v_c(n)}{\partial v_{lc}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\}$$

$$H(3N+n, 1) = \frac{\partial i_a(n)}{\partial G} = v_{ra}(n) \quad \forall n \in \{1, 2, \ldots, N\}$$

$$H(4N+n, 1) = \frac{\partial i_b(n)}{\partial G} = v_{rb}(n) \quad \forall n \in \{1, 2, \ldots, N\}$$

$$H(5N+n, 1) = \frac{\partial i_c(n)}{\partial G} = v_{rc}(n) \quad \forall n \in \{1, 2, \ldots, N\}$$

$$H(3N+n, 3+n) = \frac{\partial i_a(n)}{\partial v_{ra}(n)} = G \quad \forall n \in \{1, 2, \ldots, N\}$$

$$H(4N+n, 3+N+n) = \frac{\partial i_b(n)}{\partial v_{rb}(n)} = G \quad \forall n \in \{1, 2, \ldots, N\}$$

$$H(5N+n, 2+2N+n) = \frac{\partial i_c(n)}{\partial v_{rc}(n)} = G \quad \forall n \in \{1, 2, \ldots, N\}$$

$$H(6N+n-2, 1) = \frac{\partial z_b(n-2)}{\partial G} = v_{rb}(n) - v_{rb}(n-2) \quad \forall n \in \{3, 4, \ldots, N\}$$

$$H(7N+n-2, 1) = \frac{\partial z_c(n-2)}{\partial G} = v_{rc}(n) - v_{rc}(n-2) \quad \forall n \in \{3, 4, \ldots, N\}$$

$$H(6N+n-2, 2+N+n) = \frac{\partial z_b(n-2)}{\partial v_{rb}(n)} = G \quad \forall n \in \{3, 4, \ldots, N\}$$

$$H(7N+n-2, 2+2N+n) = \frac{\partial z_c(n-2)}{\partial v_{rc}(n)} = G \quad \forall n \in \{3, 4, \ldots, N\}$$

$$
\begin{aligned}
H(6N+n-2,N+n) &= \frac{\partial z_b(n-2)}{\partial v_{rb}(n-2)} = -G \quad \forall n \in \{3,4,\ldots,N\} \\
H(7N+n-2,2N+n) &= \frac{\partial z_c(n-2)}{\partial v_{rc}(n-2)} = -G \quad \forall n \in \{3,4,\ldots,N\} \\
&\forall n \in \{3,4\ldots,N\} \\
H(6N+n-2,2) &= \frac{\partial z_b(n-2)}{\partial \Lambda} = -\frac{\Delta t}{3}(v_{lb}(n)+4v_{lb}(n-1)+v_{lb}(n-2)) \\
&\forall n \in \{3,4\ldots,N\} \\
H(7N+n-2,2) &= \frac{\partial z_c(n-2)}{\partial \Lambda} = -\frac{\Delta t}{3}(v_{lc}(n)+4v_{lc}(n-1)+v_{lc}(n-2)) \\
&\forall n \in \{3,4\ldots,N\} \\
H(6N+n-2,3+3N+n) &= \frac{\partial z_b(n-2)}{\partial v_{lb}(n)} = -\frac{\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(7N+n-2,3+4N+n) &= \frac{\partial z_c(n-2)}{\partial v_{lc}(n)} = -\frac{\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(6N+n-2,2+3N+n) &= \frac{\partial z_b(n-2)}{\partial v_{lb}(n-1)} = -\frac{4\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(7N+n-2,2+4N+n) &= \frac{\partial z_c(n-2)}{\partial v_{lc}(n-1)} = -\frac{4\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(6N+n-2,1+3N+n) &= \frac{\partial z_b(n-2)}{\partial v_{lb}(n-2)} = -\frac{\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(7N+n-2,1+4N+n) &= \frac{\partial z_c(n-2)}{\partial v_{lc}(n-2)} = -\frac{\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\}. \qquad (72)
\end{aligned}
$$

3.4 Grounded-Wye Load with Line-Line Fault

The sampled output of the system illustrated in Fig. 9 is

$$
\mathbf{y} = \begin{bmatrix}\mathbf{v}_a & \mathbf{v}_b & \mathbf{v}_c & \mathbf{i}_a & \mathbf{i}_b & \mathbf{i}_c, & \mathbf{z}_a & \mathbf{z}_b & \mathbf{z}_c\end{bmatrix}^T \qquad (73)
$$

The output state-mapping for the system is given by

$$
v_\phi(n) = v_{r\phi}(n) + v_{l\phi}(n) \quad \forall n \in \{1,2,\ldots,N\}, \phi \in \{a,b,c\},
$$

$$
i_\phi(n) = \begin{cases} Gv_{ra}(n) + G_f(v_{ra}(n)+v_{la}(n)-v_{rb}(n)-v_{lb}(n)) \quad \forall n \in \{1,2,\ldots,N\} & \phi = a \\ Gv_{rb}(n) - G_f(v_{ra}(n)+v_{la}(n)-v_{rb}(n)-v_{lb}(n)) \quad \forall n \in \{1,2,\ldots,N\} & \phi = b \\ Gv_{rc}(n) \quad \forall \in [1,2,\ldots,N] & \phi = c \end{cases}
$$

$$
\begin{aligned}
z_\phi(n-2) = G(v_{r\phi}(n) - v_{r\phi}(n-2)) - \frac{2\Delta t\Lambda}{6}(v_{l\phi}(n)+4v_{l\phi}(n-1)+v_{l\phi}(n-2)) \\
\forall \phi \in \{b,c\}, n \in \{3,4\ldots,N\}. \qquad (74)
\end{aligned}
$$

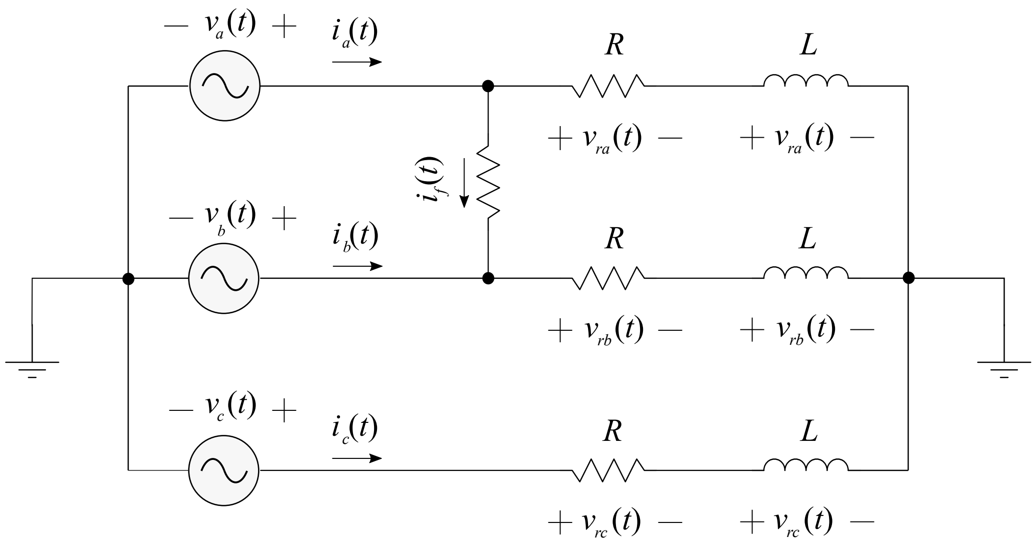


Figure 9: Dynamic model for a grounded-wye-connected RL load with a line-line fault

The output function $\mathbf{h}(\mathbf{x})$ can be written as

$$
\begin{aligned}
h_n(\mathbf{x}) &= v_{ra}(n) + v_{la}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+N}(\mathbf{x}) &= v_{rb}(n) + v_{lb}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+2N}(\mathbf{x}) &= v_{rb}(n) + v_{lb}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+3N}(\mathbf{x}) &= Gv_{ra}(n) + G_f(v_{ra}(n) + v_{la}(n) - v_{rb}(n) - v_{lb}(n)) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+4N}(\mathbf{x}) &= Gv_{rb}(n) - G_f(v_{ra}(n) + v_{la}(n) - v_{rb}(n) - v_{lb}(n)) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+5N}(\mathbf{x}) &= Gv_{rc}(n)) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+6N}(\mathbf{x}) &= G(v_{ra}(n) - v_{ra}(n-2)) - \frac{2\Delta t\Lambda}{6}(v_{la}(n) + 4v_{la}(n-1) + v_{la}(n-2)) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+7N}(\mathbf{x}) &= G(v_{rb}(n) - v_{rb}(n-2)) - \frac{2\Delta t\Lambda}{6}(v_{lb}(n) + 4v_{lc}(n-1) + v_{lb}(n-2)) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+8N}(\mathbf{x}) &= G(v_{rc}(n) - v_{rc}(n-2)) - \frac{2\Delta t\Lambda}{6}(v_{lc}(n) + 4v_{lc}(n-1) + v_{lc}(n-2)) \quad \forall n \in \{1, 2, \ldots, N\}
\end{aligned}
\tag{75}
$$

Given the variable and state vector mapping, the Jacobian can be built as follows

$$
\begin{aligned}
H(n,3+n) &= \frac{\partial v_a(n)}{\partial v_{ra}(n)} = 1 \quad \forall n \in \{1,2,\ldots,N\} \\
H(N+n,3+N+n) &= \frac{\partial v_b(n)}{\partial v_{rb}(n)} = 1 \quad \forall n \in \{1,2,\ldots,N\} \\
H(2N+n,3+2N+n) &= \frac{\partial v_c(n)}{\partial v_{rc}(n)} = 1 \quad \forall n \in \{1,2,\ldots,N\} \\
H(n,3+3N+n) &= \frac{\partial v_a(n)}{\partial v_{la}(n)} = 1 \quad \forall n \in \{1,2,\ldots,N\} \\
H(N+n,3+4N+n) &= \frac{\partial v_b(n)}{\partial v_{lb}(n)} = 1 \quad \forall n \in \{1,2,\ldots,N\} \\
H(2N+n,3+5N+n) &= \frac{\partial v_c(n)}{\partial v_{lc}(n)} = 1 \quad \forall n \in \{1,2,\ldots,N\} \\
H(3N+n,1) &= \frac{\partial i_a(n)}{\partial G_f} = v_{ra}(n) \quad \forall n \in \{1,2,\ldots,N\} \\
H(4N+n,1) &= \frac{\partial i_b(n)}{\partial G_f} = v_{rb}(n) \quad \forall n \in \{1,2,\ldots,N\} \\
H(5N+n,1) &= \frac{\partial i_c(n)}{\partial G} = v_{rc}(n) \quad \forall n \in \{1,2,\ldots,N\} \\
H(3N+n,3) &= \frac{\partial i_a(n)}{\partial G_f} = (v_{ra}(n)+v_{la}(n)) - (v_{rb}(n)+v_{rb}(n)) \quad \forall n \in \{1,2,\ldots,N\} \\
H(4N+n,3) &= \frac{\partial i_b(n)}{\partial G_f} = -(v_{ra}(n)+v_{la}(n)) + (v_{rb}(n)+v_{rb}(n)) \quad \forall n \in \{1,2,\ldots,N\} \\
H(3N+n,3+n) &= \frac{\partial i_a(n)}{\partial v_{ra}(n)} = G+G_f \quad \forall n \in \{1,2,\ldots,N\} \\
H(3N+n,3+3N+n) &= \frac{\partial i_a(n)}{\partial v_{la}(n)} = G_f \quad \forall n \in \{1,2,\ldots,N\} \\
H(3N+n,3+N+n) &= \frac{\partial i_a(n)}{\partial v_{rb}(n)} = -G_f \quad \forall n \in \{1,2,\ldots,N\} \\
H(3N+n,3+4N+n) &= \frac{\partial i_a(n)}{\partial v_{lb}(n)} = -G_f \quad \forall n \in \{1,2,\ldots,N\}
\end{aligned}
\tag{76}
$$

$$H(4N+n,3+n)=\frac{\partial i_b(n)}{\partial v_{ra}(n)}=-G_f \quad \forall n\in\{1,2,\dots,N\}$$

$$H(4N+n,3+3N+n)=\frac{\partial i_b(n)}{\partial v_{la}(n)}=-G_f \quad \forall n\in\{1,2,\dots,N\}$$

$$H(4N+n,3+N+n)=\frac{\partial i_b(n)}{\partial v_{rb}(n)}=G+G_f \quad \forall n\in\{1,2,\dots,N\}$$

$$H(4N+n,3+4N+n)=\frac{\partial i_b(n)}{\partial v_{lb}(n)}=G_f \quad \forall n\in\{1,2,\dots,N\}$$

$$H(4N+n,3+N+n)=\frac{\partial i_b(n)}{\partial v_{rb}(n)}=G \quad \forall n\in\{1,2,\dots,N\}$$

$$H(5N+n,3+2N+n)=\frac{\partial i_c(n)}{\partial v_{rc}(n)}=G \quad \forall n\in\{1,2,\dots,N\}$$

$$H(6N+n-2,1)=\frac{\partial z_a(n-2)}{\partial G}=v_{ra}(n)-v_{ra}(n-2) \quad \forall n\in\{3,4,\dots,N\}$$

$$H(7N+n-2,1)=\frac{\partial z_b(n-2)}{\partial G}=v_{rb}(n)-v_{rb}(n-2) \quad \forall n\in\{3,4,\dots,N\}$$

$$H(8N+n-2,1)=\frac{\partial z_c(n-2)}{\partial G}=v_{rc}(n)-v_{rc}(n-2) \quad \forall n\in\{3,4,\dots,N\}$$

$$H(6N+n-2,3+n)=\frac{\partial z_a(n-2)}{\partial v_{ra}(n)}=G \quad \forall n\in\{3,4,\dots,N\}$$

$$H(7N+n-2,3+N+n)=\frac{\partial z_b(n-2)}{\partial v_{rb}(n)}=G \quad \forall n\in\{3,4,\dots,N\}$$

$$H(8N+n-2,3+2N+n)=\frac{\partial z_c(n-2)}{\partial v_{rc}(n)}=G \quad \forall n\in\{3,4,\dots,N\}$$

$$H(6N+n-2,n)=\frac{\partial z_a(n-2)}{\partial v_{ra}(n-2)}=-G \quad \forall n\in\{3,4,\dots,N\}$$

$$H(7N+n-2,N+n)=\frac{\partial z_b(n-2)}{\partial v_{rb}(n-2)}=-G \quad \forall n\in\{3,4,\dots,N\}$$

$$H(8N+n-2,2N+n)=\frac{\partial z_c(n-2)}{\partial v_{rc}(n-2)}=-G \quad \forall n\in\{3,4,\dots,N\}$$

$$H(6N+n-2,2)=\frac{\partial z_a(n-2)}{\partial \Lambda}=-\frac{\Delta t}{3}(v_{la}(n)+4v_{la}(n-1)+v_{la}(n-2)) \quad \forall n\in\{3,4\dots,N\}$$

$$H(7N+n-2,2)=\frac{\partial z_b(n-2)}{\partial \Lambda}=-\frac{\Delta t}{3}(v_{lb}(n)+4v_{lb}(n-1)+v_{lb}(n-2)) \quad \forall n\in\{3,4\dots,N\}$$

$$H(8N+n-2,2)=\frac{\partial z_c(n-2)}{\partial \Lambda}=-\frac{\Delta t}{3}(v_{lc}(n)+4v_{lc}(n-1)+v_{lc}(n-2)) \quad \forall n\in\{3,4\dots,N\}$$

$$H(6N+n-2,2+3N+n)=\frac{\partial z_a(n-2)}{\partial v_{la}(n)}=-\frac{\Delta t\Lambda}{3} \quad \forall n\in\{3,4\dots,N\}$$

$$
\begin{aligned}
H(7N+n-2, 2+4N+n) &= \frac{\partial z_b(n-2)}{\partial v_{lb}(n)} = -\frac{\Delta t \Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(8N+n-2, 2+5N+n) &= \frac{\partial z_c(n-2)}{\partial v_{lc}(n)} = -\frac{\Delta t \Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(6N+n-2, 1+3N+n) &= \frac{\partial z_a(n-2)}{\partial v_{la}(n-1)} = -\frac{4\Delta t \Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(6N+n-2, 1+4N+n) &= \frac{\partial z_b(n-2)}{\partial v_{lb}(n-1)} = -\frac{4\Delta t \Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(6N+n-2, 1+5N+n) &= \frac{\partial z_c(n-2)}{\partial v_{lc}(n-1)} = -\frac{4\Delta t \Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(6N+n-2, 3N+n) &= \frac{\partial z_a(n-2)}{\partial v_{la}(n-2)} = -\frac{\Delta t \Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(7N+n-2, 4N+n) &= \frac{\partial z_b(n-2)}{\partial v_{lb}(n-2)} = -\frac{\Delta t \Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(8N+n-2, 5N+n) &= \frac{\partial z_c(n-2)}{\partial v_{lc}(n-2)} = -\frac{\Delta t \Lambda}{3} \quad \forall n \in \{3,4\ldots,N\}. \quad (77)
\end{aligned}
$$

The state of the system can then be solved for by applying eqs 14–16.

3.5 Delta Load without Fault

The sampled output of the system illustrated in Fig. 11 is

$$
\mathbf{y} = \begin{bmatrix} \mathbf{v}_{ab} & \mathbf{v}_{bc} & \mathbf{v}_{ca} & \mathbf{i}_a & \mathbf{i}_b & \mathbf{i}_c, & \mathbf{z}_{ab} & \mathbf{z}_{bc} & \mathbf{z}_{ca} \end{bmatrix}^T. \quad (78)
$$

The state for the system is

$$
x(t) = \begin{bmatrix} G & \Lambda, \mathbf{v}_{rab} & \mathbf{v}_{rbc} & \mathbf{v}_{rca} & \mathbf{v}_{lab} & \mathbf{v}_{lbc} & \mathbf{v}_{lca} \end{bmatrix}^T. \quad (79)
$$

The output state-mapping for the system is given by

$$
\begin{aligned}
& v_\psi(n) = v_{r\psi}(n) + v_{l\psi}(n) \quad \forall n \in \{1,2,\ldots,N\}, \phi \in \{ab, bc, ca\}, \\
& i_a(n) = G(v_{rab}(n) - v_{rca}(n)) \\
& i_b(n) = G(v_{rbc}(n) - v_{rab}(n)) \\
& i_c(n) = G(v_{rca}(n) - v_{rbc}(n))
\end{aligned}
$$

$$
\begin{aligned}
z_\psi(n-2) = G(v_{r\psi}(n) - v_{r\psi}(n-2)) - \frac{2\Delta t \Lambda}{6}(v_{l\psi}(n) + 4v_{l\psi}(n-1) + v_{l\psi}(n-2)) \\
\forall \psi \in \{ab, bc, ca\}, n \in \{3,4\ldots,N\}. \quad (80)
\end{aligned}
$$

The output function $\mathbf{h}(\mathbf{x})$ can be written as

$$
\begin{aligned}
h_n(\mathbf{x}) &= v_{rab}(n) + v_{lab}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+N}(\mathbf{x}) &= v_{rbc}(n) + v_{lbc}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+2N}(\mathbf{x}) &= v_{rca}(n) + v_{lca}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+3N}(\mathbf{x}) &= G(v_{rab}(n) - v_{rca}(n)) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+4N}(\mathbf{x}) &= G(v_{rbc}(n) - v_{rab}(n)) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+5N}(\mathbf{x}) &= G(v_{rca}(n) - v_{rbc}(n)) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+6N}(\mathbf{x}) &= G(v_{rab}(n) - v_{rab}(n-2)) - \frac{2\Delta t\Lambda}{6}(v_{lab}(n) + 4v_{lab}(n-1) + v_{lab}(n-2)) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+7N}(\mathbf{x}) &= G(v_{rbc}(n) - v_{rbc}(n-2)) - \frac{2\Delta t\Lambda}{6}(v_{lbc}(n) + 4v_{lbc}(n-1) + v_{lbc}(n-2)) \quad \forall n \in \{1, 2, \ldots, N\} \\
h_{n+8N}(\mathbf{x}) &= G(v_{rca}(n) - v_{rca}(n-2)) - \frac{2\Delta t\Lambda}{6}(v_{lca}(n) + 4v_{lca}(n-1) + v_{lca}(n-2)) \quad \forall n \in \{1, 2, \ldots, N\}
\end{aligned} \tag{81}
$$

Given the variable and state vector mapping, the Jacobian can be built as follows

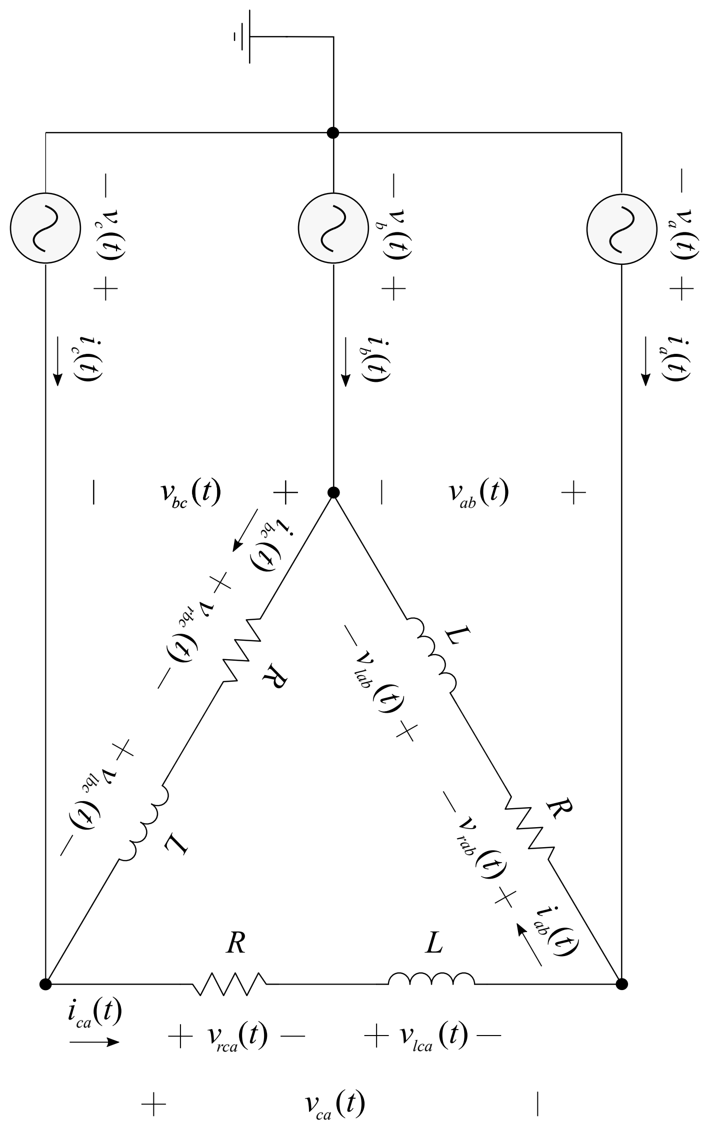


Figure 10: Dynamic model for a delta-connected RL load

$$
\begin{aligned}
H(n, 2+n) &= \frac{\partial v_{ab}(n)}{\partial v_{rab}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\} \\
H(N+n, 2+N+n) &= \frac{\partial v_{bc}(n)}{\partial v_{rbc}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\} \\
H(2N+n, 2+2N+n) &= \frac{\partial v_{ca}(n)}{\partial v_{rca}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\} \\
H(n, 2+3N+n) &= \frac{\partial v_{ab}(n)}{\partial v_{lab}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\} \\
H(N+n, 2+4N+n) &= \frac{\partial v_{bc}(n)}{\partial v_{lbc}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\} \\
H(2N+n, 2+5N+n) &= \frac{\partial v_{ca}(n)}{\partial v_{lca}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\} \\
H(3N+n, 1) &= \frac{\partial i_a(n)}{\partial G} = v_{rab}(n) - v_{rca}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
H(4N+n, 1) &= \frac{\partial i_b(n)}{\partial G} = v_{rbc}(n) - v_{rab}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
H(5N+n, 1) &= \frac{\partial i_c(n)}{\partial G} = v_{rca}(n) - v_{rbc}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
H(3N+n, 2+n) &= \frac{\partial i_a(n)}{\partial v_{rab}(n)} = G \quad \forall n \in \{1, 2, \ldots, N\} \\
H(3N+n, 2+2N+n) &= \frac{\partial i_a(n)}{\partial v_{rca}(n)} = -G \quad \forall n \in \{1, 2, \ldots, N\} \\
H(4N+n, 2+N+n) &= \frac{\partial i_b(n)}{\partial v_{rbc}(n)} = G \quad \forall n \in \{1, 2, \ldots, N\} \\
H(4N+n, 2+n) &= \frac{\partial i_b(n)}{\partial v_{rab}(n)} = -G \quad \forall n \in \{1, 2, \ldots, N\} \\
H(5N+n, 2+2N+n) &= \frac{\partial i_c(n)}{\partial v_{rca}(n)} = G \quad \forall n \in \{1, 2, \ldots, N\} \\
H(5N+n, 2+N+n) &= \frac{\partial i_c(n)}{\partial v_{rbc}(n)} = -G \quad \forall n \in \{1, 2, \ldots, N\} \\
H(6N+n-2, 1) &= \frac{\partial z_{ab}(n-2)}{\partial G} = v_{rab}(n) - v_{rab}(n-2) \quad \forall n \in \{3, 4, \ldots, N\} \\
H(7N+n-2, 1) &= \frac{\partial z_{bc}(n-2)}{\partial G} = v_{rbc}(n) - v_{rbc}(n-2) \quad \forall n \in \{3, 4, \ldots, N\} \\
H(8N+n-2, 1) &= \frac{\partial z_{ca}(n-2)}{\partial G} = v_{rca}(n) - v_{rca}(n-2) \quad \forall n \in \{3, 4, \ldots, N\}
\end{aligned}
\tag{82}
$$

$$
\begin{aligned}
H(6N+n-2,2+n) &= \frac{\partial z_{ab}(n-2)}{\partial v_{rab}(n)} = G \quad \forall n \in \{3,4,\ldots,N\} \\
H(7N+n-2,2+N+n) &= \frac{\partial z_{bc}(n-2)}{\partial v_{rbc}(n)} = G \quad \forall n \in \{3,4,\ldots,N\} \\
H(8N+n-2,2+2N+n) &= \frac{\partial z_{ca}(n-2)}{\partial v_{rca}(n)} = G \quad \forall n \in \{3,4,\ldots,N\} \\
H(6N+n-2,n) &= \frac{\partial z_{ab}(n-2)}{\partial v_{rab}(n-2)} = -G \quad \forall n \in \{3,4,\ldots,N\} \\
H(7N+n-2,N+n) &= \frac{\partial z_{bc}(n-2)}{\partial v_{rbc}(n-2)} = -G \quad \forall n \in \{3,4,\ldots,N\} \\
H(8N+n-2,2N+n) &= \frac{\partial z_{ca}(n-2)}{\partial v_{rca}(n-2)} = -G \quad \forall n \in \{3,4,\ldots,N\} \\
H(6N+n-2,2) &= \frac{\partial z_{ab}(n-2)}{\partial \Lambda} = -\frac{\Delta t}{3}(v_{lab}(n)+4v_{lab}(n-1)+v_{lab}(n-2)) \\
&\forall n \in \{3,4\ldots,N\} \\
H(7N+n-2,2) &= \frac{\partial z_{bc}(n-2)}{\partial \Lambda} = -\frac{\Delta t}{3}(v_{lbc}(n)+4v_{lbc}(n-1)+v_{lbc}(n-2)) \\
&\forall n \in \{3,4\ldots,N\} \\
H(8N+n-2,2) &= \frac{\partial z_{ca}(n-2)}{\partial \Lambda} = -\frac{\Delta t}{3}(v_{lca}(n)+4v_{lca}(n-1)+v_{lca}(n-2)) \\
&\forall n \in \{3,4\ldots,N\} \\
H(6N+n-2,2+3N+n) &= \frac{\partial z_{ab}(n-2)}{\partial v_{lab}(n)} = -\frac{\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(7N+n-2,2+4N+n) &= \frac{\partial z_{bc}(n-2)}{\partial v_{lbc}(n)} = -\frac{\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(8N+n-2,2+5N+n) &= \frac{\partial z_{ca}(n-2)}{\partial v_{lca}(n)} = -\frac{\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(6N+n-2,1+3N+n) &= \frac{\partial z_{ab}(n-2)}{\partial v_{lab}(n-1)} = -\frac{4\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(6N+n-2,1+4N+n) &= \frac{\partial z_{bc}(n-2)}{\partial v_{lbc}(n-1)} = -\frac{4\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(6N+n-2,1+5N+n) &= \frac{\partial z_{ca}(n-2)}{\partial v_{lca}(n-1)} = -\frac{4\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(6N+n-2,3N+n) &= \frac{\partial z_{ab}(n-2)}{\partial v_{lab}(n-2)} = -\frac{\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(7N+n-2,4N+n) &= \frac{\partial z_{bc}(n-2)}{\partial v_{lbc}(n-2)} = -\frac{\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(8N+n-2,5N+n) &= \frac{\partial z_{ca}(n-2)}{\partial v_{lca}(n-2)} = -\frac{\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\}.
\end{aligned} \tag{83}
$$

The state of the system can then be solved for by applying eqs 14–16.

3.6 Delta Load with Line-Line Fault

The sampled output of the system illustrated in Fig. 9 is

$$\mathbf{y} = \begin{bmatrix}\mathbf{v}_{ab} & \mathbf{v}_{bc} & \mathbf{v}_{ca} & \mathbf{i}_a & \mathbf{i}_b & \mathbf{i}_c, & \mathbf{z}_{bc} & \mathbf{z}_{ca}\end{bmatrix}^T. \tag{84}$$

Note that there are no $z_{ab}(n)$ output variables as the reactive impedance on phase a is large compared to the parallel fault conductance G_f. The state for the system is

$$x(t) = \begin{bmatrix}G & \Lambda & G_f & \mathbf{v}_{rab} & \mathbf{v}_{rbc} & \mathbf{v}_{rca} & \mathbf{v}_{lbc} & \mathbf{v}_{lca}\end{bmatrix}^T. \tag{85}$$

where $G_f = R^{-1}$ is the conductance and the remaining states are the same as those in that of the delta no-fault state in eq. 79. The output state-mapping for the system is given by

$$v_\psi(n) = \begin{cases} V_{r\psi}(n) \quad \forall \in [1,2,\ldots,N] & \psi = ab, \\ v_{r\psi}(n) + v_{l\psi}(n) \quad \forall \in [1,2,\ldots,N] & \psi \in \{b,c\}, \end{cases}$$

$$\begin{aligned} i_a(n) &= G_f v_{rab}(n) - G v_{rca}(n) \\ i_b(n) &= G v_{rbc}(n) - G_f v_{rab}(n) \\ i_c(n) &= G(v_{rca}(n) - v_{rbc}(n)) \end{aligned}$$

$$\begin{aligned} z_\psi(n-2) = G(v_{r\psi}(n) - v_{r\psi}(n-2)) - \frac{2\Delta t\Lambda}{6}(v_{l\psi}(n) + 4v_{l\psi}(n-1) + v_{l\psi}(n-2)) \\ \forall \psi \in \{bc, ca\}, n \in \{3,4\ldots,N\} \end{aligned} \tag{86}$$

$$\begin{aligned} h_n(\mathbf{x}) &= v_{rab}(n) \quad \forall n \in \{1,2,\ldots,N\} \\ h_{n+N}(\mathbf{x}) &= v_{rbc}(n) + v_{lbc}(n) \quad \forall n \in \{1,2,\ldots,N\} \\ h_{n+2N}(\mathbf{x}) &= v_{rca}(n) + v_{lca}(n) \quad \forall n \in \{1,2,\ldots,N\} \\ h_{n+3N}(\mathbf{x}) &= G v_{ra}(n) \quad \forall n \in \{1,2,\ldots,N\} \\ h_{n+4N}(\mathbf{x}) &= G v_{rb}(n) \quad \forall n \in \{1,2,\ldots,N\} \\ h_{n+5N}(\mathbf{x}) &= G v_{rc}(n)) \quad \forall n \in \{1,2,\ldots,N\} \\ h_{n+6N}(\mathbf{x}) &= G(v_{rbc}(n) - v_{rbc}(n-2)) - \frac{2\Delta t\Lambda}{6}(v_{lbc}(n) + 4v_{lbc}(n-1) + v_{lbc}(n-2)) \quad \forall n \in \{1,2,\ldots,N\} \\ h_{n+7N}(\mathbf{x}) &= G(v_{rca}(n) - v_{rca}(n-2)) - \frac{2\Delta t\Lambda}{6}(v_{lca}(n) + 4v_{lca}(n-1) + v_{lca}(n-2)) \quad \forall n \in \{1,2,\ldots,N\}. \end{aligned} \tag{87}$$

Given the variable and state vector mapping, the Jacobian can be built as follows

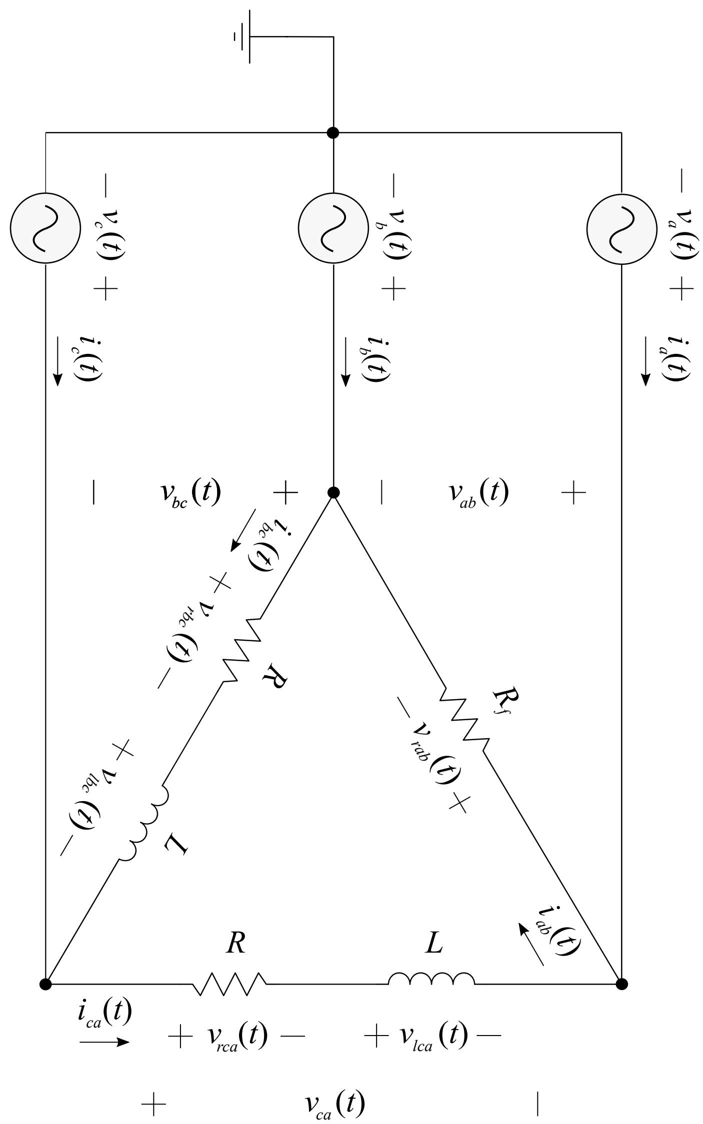


Figure 11: Dynamic model for a delta-connected RL load with a line-line fault

$$H(n,3+n)=\frac{\partial v_{ab}(n)}{\partial v_{rab}(n)}=1 \quad \forall n\in\{1,2,\dots,N\}$$
$$H(N+n,3+N+n)=\frac{\partial v_{bc}(n)}{\partial v_{rbc}(n)}=1 \quad \forall n\in\{1,2,\dots,N\}$$
$$H(2N+n,3+2N+n)=\frac{\partial v_{ca}(n)}{\partial v_{rca}(n)}=1 \quad \forall n\in\{1,2,\dots,N\}$$
$$H(N+n,3+3N+n)=\frac{\partial v_{bc}(n)}{\partial v_{lbc}(n)}=1 \quad \forall n\in\{1,2,\dots,N\}$$
$$H(2N+n,3+4N+n)=\frac{\partial v_{ca}(n)}{\partial v_{lca}(n)}=1 \quad \forall n\in\{1,2,\dots,N\}$$
$$H(3N+n,1)=\frac{\partial i_a(n)}{\partial G}=v_{rca}(n) \quad \forall n\in\{1,2,\dots,N\}$$
$$H(4N+n,1)=\frac{\partial i_b(n)}{\partial G}=v_{rbc}(n) \quad \forall n\in\{1,2,\dots,N\}$$
$$H(5N+n,1)=\frac{\partial i_c(n)}{\partial G}=v_{rca}(n)-v_{rbc}(n) \quad \forall n\in\{1,2,\dots,N\}$$
$$H(3N+n,3)=\frac{\partial i_a(n)}{\partial G_f}=v_{rab}(n) \quad \forall n\in\{1,2,\dots,N\}$$
$$H(4N+n,3)=\frac{\partial i_b(n)}{\partial G_F}=-v_{rab}(n) \quad \forall n\in\{1,2,\dots,N\}$$
$$H(3N+n,3+n)=\frac{\partial i_a(n)}{\partial v_{rab}(n)}=G_f \quad \forall n\in\{1,2,\dots,N\}$$
$$H(3N+n,3+N+n)=\frac{\partial i_a(n)}{\partial v_{rbc}(n)}=-G \quad \forall n\in\{1,2,\dots,N\}$$
$$H(4N+n,3+n)=\frac{\partial i_b(n)}{\partial v_{rab}(n)}=-G_f \quad \forall n\in\{1,2,\dots,N\}$$
$$H(4N+n,3+N+n)=\frac{\partial i_b(n)}{\partial v_{rbc}(n)}=G \quad \forall n\in\{1,2,\dots,N\}$$
$$H(5N+n,2+2+N+n)=\frac{\partial i_c(n)}{\partial v_{rbc}(n)}=-G \quad \forall n\in\{1,2,\dots,N\}$$
$$H(5N+n,3+2N+n)=\frac{\partial i_c(n)}{\partial v_{rca}(n)}=G \quad \forall n\in\{1,2,\dots,N\}$$
$$H(6N+n-2,1)=\frac{\partial z_{bc}(n-2)}{\partial G}=v_{rbc}(n)-v_{rbc}(n-2) \quad \forall n\in\{3,4,\dots,N\}$$
$$H(7N+n-2,1)=\frac{\partial z_{ca}(n-2)}{\partial G}=v_{rca}(n)-v_{rca}(n-2) \quad \forall n\in\{3,4,\dots,N\}$$
$$H(6N+n-2,2+N+n)=\frac{\partial z_{bc}(n-2)}{\partial v_{rbc}(n)}=G \quad \forall n\in\{3,4,\dots,N\}$$
$$H(7N+n-2,2+2N+n)=\frac{\partial z_{ca}(n-2)}{\partial v_{rca}(n)}=G \quad \forall n\in\{3,4,\dots,N\}$$

$$
\begin{aligned}
H(6N+n-2, N+n) &= \frac{\partial z_{bc}(n-2)}{\partial v_{rbc}(n-2)} = -G \quad \forall n \in \{3, 4, \ldots, N\} \\
H(7N+n-2, 2N+n) &= \frac{\partial z_{ca}(n-2)}{\partial v_{rca}(n-2)} = -G \quad \forall n \in \{3, 4, \ldots, N\} \\
H(6N+n-2, 2) &= \frac{\partial z_{bc}(n-2)}{\partial \Lambda} = -\frac{\Delta t}{3}(v_{lbc}(n) + 4v_{lbc}(n-1) + v_{lbc}(n-2)) \\
&\quad \forall n \in \{3, 4 \ldots, N\} \\
H(7N+n-2, 2) &= \frac{\partial z_{ca}(n-2)}{\partial \Lambda} = -\frac{\Delta t}{3}(v_{lca}(n) + 4v_{lca}(n-1) + v_{lca}(n-2)) \\
&\quad \forall n \in \{3, 4 \ldots, N\} \\
H(6N+n-2, 3+3N+n) &= \frac{\partial z_{bc}(n-2)}{\partial v_{lbc}(n)} = -\frac{\Delta t \Lambda}{3} \quad \forall n \in \{3, 4 \ldots, N\} \\
H(7N+n-2, 3+4N+n) &= \frac{\partial z_{ca}(n-2)}{\partial v_{lca}(n)} = -\frac{\Delta t \Lambda}{3} \quad \forall n \in \{3, 4 \ldots, N\} \\
H(6N+n-2, 2+3N+n) &= \frac{\partial z_{bc}(n-2)}{\partial v_{lbc}(n-1)} = -\frac{4\Delta t \Lambda}{3} \quad \forall n \in \{3, 4 \ldots, N\} \\
H(7N+n-2, 2+4N+n) &= \frac{\partial z_{ca}(n-2)}{\partial v_{lca}(n-1)} = -\frac{4\Delta t \Lambda}{3} \quad \forall n \in \{3, 4 \ldots, N\} \\
H(6N+n-2, 1+3N+n) &= \frac{\partial z_{bc}(n-2)}{\partial v_{lbc}(n-2)} = -\frac{\Delta t \Lambda}{3} \quad \forall n \in \{3, 4 \ldots, N\} \\
H(7N+n-2, 1+4N+n) &= \frac{\partial z_{ca}(n-2)}{\partial v_{lca}(n-2)} = -\frac{\Delta t \Lambda}{3} \quad \forall n \in \{3, 4 \ldots, N\}.
\end{aligned} \tag{88}
$$

The state of the system can then be solved for by applying eqs 14–16.

3.7 Delta Load with Line-Ground Fault

The sampled output of the system illustrated in Fig. 12 is

$$
\mathbf{y} = \begin{bmatrix} \mathbf{v}_{ab} & \mathbf{v}_{bc} & \mathbf{v}_{ca} & \mathbf{v}_{a} & \mathbf{i}_{a} & \mathbf{i}_{b} & \mathbf{i}_{c}, & \mathbf{z}_{ab} & \mathbf{z}_{bc} & \mathbf{z}_{ca} \end{bmatrix}^T. \tag{89}
$$

The state for the system is

$$
x(t) = \begin{bmatrix} G & \Lambda & G_f & \mathbf{v}_{rab} & \mathbf{v}_{rbc} & \mathbf{v}_{rca} & \mathbf{v}_{lab} & \mathbf{v}_{lbc} & \mathbf{v}_{lca} & \mathbf{v}_{f} \end{bmatrix}^T. \tag{90}
$$

where $G_f = R^{-1}$ is the fault conductance, $\mathbf{v}_f$ is the voltage across the fault and the remaining states are the same as those in that of the delta no-fault state in eq. 79. The output state-mapping for the system is given by

$$\begin{aligned}
v_\psi(n) &= v_{r\psi}(n) + v_{l\psi}(n) \quad \forall n \in \{1,2,\ldots,N\}, \phi \in \{ab, bc, ca\}, \\
v_a(n) &= v_f(n) \qquad (91) \\
i_a(n) &= G(v_{rab}(n) - v_{rca}(n)) + G_f v_f(n) \\
i_b(n) &= G(v_{rbc}(n) - v_{rab}(n)) \\
i_c(n) &= G(v_{rca}(n) - v_{rbc}(n))
\end{aligned} \qquad (92)$$

$$\begin{aligned}
z_\psi(n-2) &= G(v_{r\psi}(n) - v_{r\psi}(n-2)) - \frac{2\Delta t \Lambda}{6}(v_{l\psi}(n) + 4v_{l\psi}(n-1) + v_{l\psi}(n-2)) \\
&\forall \psi \in \{ab, bc, ca\}, n \in \{3, 4\ldots, N\}.
\end{aligned} \qquad (93)$$

The output function $\mathbf{h}(\mathbf{x})$ can be written as

$$\begin{aligned}
h_n(\mathbf{x}) &= v_{rab}(n) + v_{lab}(n) \quad \forall n \in \{1,2,\ldots,N\} \\
h_{n+N}(\mathbf{x}) &= v_{rbc}(n) + v_{lbc}(n) \quad \forall n \in \{1,2,\ldots,N\} \\
h_{n+2N}(\mathbf{x}) &= v_{rca}(n) + v_{lca}(n) \quad \forall n \in \{1,2,\ldots,N\} \\
h_{n+3N}(\mathbf{x}) &= v_f(n) \quad \forall n \in \{1,2,\ldots,N\} \\
h_{n+4N}(\mathbf{x}) &= G(v_{rab}(n) - v_{rca}(n)) + G_f v_f(n) \quad \forall n \in \{1,2,\ldots,N\} \\
h_{n+5N}(\mathbf{x}) &= G(v_{rbc}(n) - v_{rab}(n)) \quad \forall n \in \{1,2,\ldots,N\} \\
h_{n+6N}(\mathbf{x}) &= G(v_{rca}(n) - v_{rbc}(n)) \quad \forall n \in \{1,2,\ldots,N\} \\
h_{n+7N}(\mathbf{x}) &= G(v_{rab}(n) - v_{rab}(n-2)) - \frac{2\Delta t \Lambda}{6}(v_{lab}(n) + 4v_{lab}(n-1) + v_{lab}(n-2)) \quad \forall n \in \{1,2,\ldots,N\} \\
h_{n+8N}(\mathbf{x}) &= G(v_{rbc}(n) - v_{rbc}(n-2)) - \frac{2\Delta t \Lambda}{6}(v_{lbc}(n) + 4v_{lbc}(n-1) + v_{lbc}(n-2)) \quad \forall n \in \{1,2,\ldots,N\} \\
h_{n+9N}(\mathbf{x}) &= G(v_{rca}(n) - v_{rca}(n-2)) - \frac{2\Delta t \Lambda}{6}(v_{lca}(n) + 4v_{lca}(n-1) + v_{lca}(n-2)) \quad \forall n \in \{1,2,\ldots,N\}.
\end{aligned} \qquad (94)$$

Given the variable and state vector mapping, the Jacobian can be built as follows

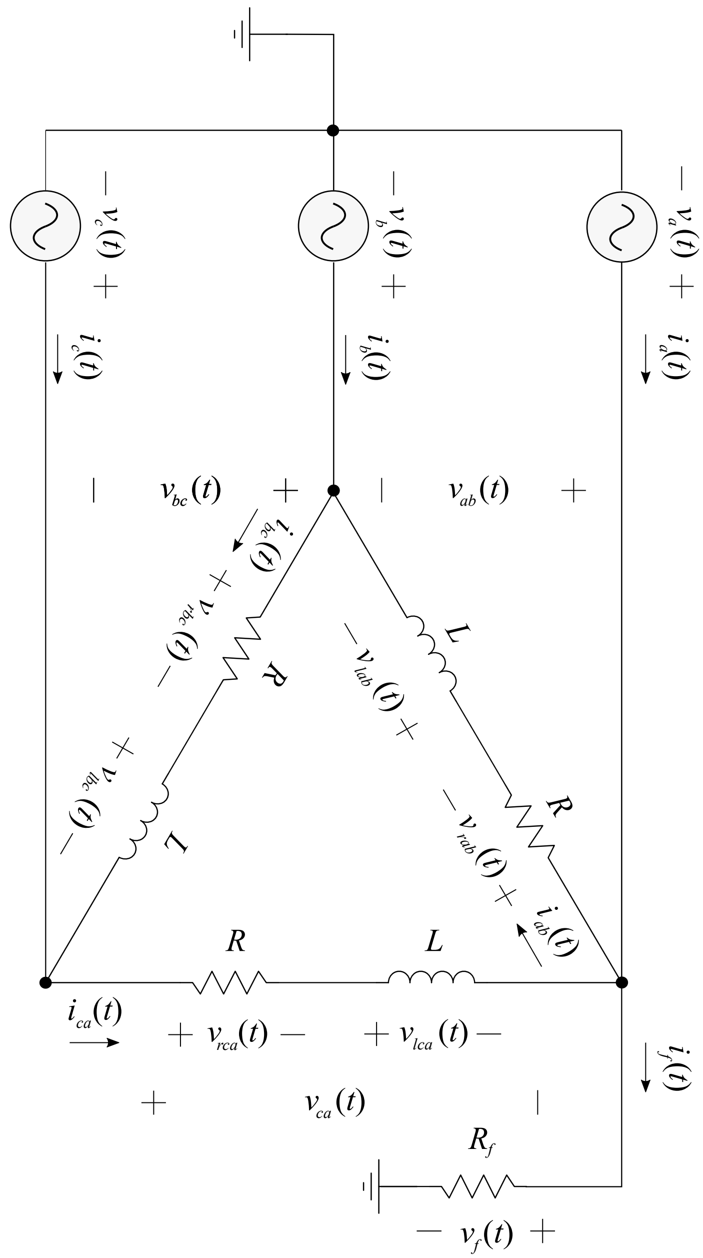


Figure 12: Dynamic model for a delta-connected RL load with a line-ground fault

$$
\begin{aligned}
H(n, 2+n) &= \frac{\partial v_{ab}(n)}{\partial v_{rab}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\} \\
H(N+n, 2+N+n) &= \frac{\partial v_{bc}(n)}{\partial v_{rbc}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\} \\
H(2N+n, 2+2N+n) &= \frac{\partial v_{ca}(n)}{\partial v_{rca}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\} \\
H(n, 2+3N+n) &= \frac{\partial v_{ab}(n)}{\partial v_{lab}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\} \\
H(N+n, 2+4N+n) &= \frac{\partial v_{bc}(n)}{\partial v_{lbc}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\} \\
H(2N+n, 2+5N+n) &= \frac{\partial v_{ca}(n)}{\partial v_{lca}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\} \\
H(3N+n, 2+6N+n) &= \frac{\partial v_{a}(n)}{\partial v_{f}(n)} = 1 \quad \forall n \in \{1, 2, \ldots, N\} \\
H(4N+n, 1) &= \frac{\partial i_{a}(n)}{\partial G} = v_{rab}(n) - v_{rca}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
H(5N+n, 1) &= \frac{\partial i_{b}(n)}{\partial G} = v_{rbc}(n) - v_{rab}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
H(6N+n, 1) &= \frac{\partial i_{c}(n)}{\partial G} = v_{rca}(n) - v_{rbc}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
H(4N+n, 3) &= \frac{\partial i_{a}(n)}{\partial G_f} = v_{f}(n) \quad \forall n \in \{1, 2, \ldots, N\} \\
H(4N+n, 2+n) &= \frac{\partial i_{a}(n)}{\partial v_{rab}(n)} = G \quad \forall n \in \{1, 2, \ldots, N\} \\
H(4N+n, 2+2N+n) &= \frac{\partial i_{a}(n)}{\partial v_{rca}(n)} = -G \quad \forall n \in \{1, 2, \ldots, N\} \\
H(4N+n, 3+6N+n) &= \frac{\partial i_{a}(n)}{\partial v_{f}(n)} = G_f \quad \forall n \in \{1, 2, \ldots, N\} \\
H(5N+n, 2+N+n) &= \frac{\partial i_{b}(n)}{\partial v_{rbc}(n)} = G \quad \forall n \in \{1, 2, \ldots, N\} \\
H(5N+n, 2+n) &= \frac{\partial i_{b}(n)}{\partial v_{rab}(n)} = -G \quad \forall n \in \{1, 2, \ldots, N\} \\
H(6N+n, 2+2N+n) &= \frac{\partial i_{c}(n)}{\partial v_{rca}(n)} = G \quad \forall n \in \{1, 2, \ldots, N\} \\
H(6N+n, 2+N+n) &= \frac{\partial i_{c}(n)}{\partial v_{rbc}(n)} = -G \quad \forall n \in \{1, 2, \ldots, N\} \\
H(7N+n-2, 1) &= \frac{\partial z_{ab}(n-2)}{\partial G} = v_{rab}(n) - v_{rab}(n-2) \quad \forall n \in \{3, 4, \ldots, N\} \\
H(8N+n-2, 1) &= \frac{\partial z_{bc}(n-2)}{\partial G} = v_{rbc}(n) - v_{rbc}(n-2) \quad \forall n \in \{3, 4, \ldots, N\} \\
H(9N+n-2, 1) &= \frac{\partial z_{ca}(n-2)}{\partial G} = v_{rca}(n) - v_{rca}(n-2) \quad \forall n \in \{3, 4, \ldots, N\}
\end{aligned}
\tag{95}
$$

$$
\begin{aligned}
H(6N+n-2, 2+n) &= \frac{\partial z_{ab}(n-2)}{\partial v_{rab}(n)} = G \quad \forall n \in \{3,4,\ldots,N\} \\
H(7N+n-2, 2+N+n) &= \frac{\partial z_{bc}(n-2)}{\partial v_{rbc}(n)} = G \quad \forall n \in \{3,4,\ldots,N\} \\
H(8N+n-2, 2+2N+n) &= \frac{\partial z_{ca}(n-2)}{\partial v_{rca}(n)} = G \quad \forall n \in \{3,4,\ldots,N\} \\
H(6N+n-2, n) &= \frac{\partial z_{ab}(n-2)}{\partial v_{rab}(n-2)} = -G \quad \forall n \in \{3,4,\ldots,N\} \\
H(7N+n-2, N+n) &= \frac{\partial z_{bc}(n-2)}{\partial v_{rbc}(n-2)} = -G \quad \forall n \in \{3,4,\ldots,N\} \\
H(8N+n-2, 2N+n) &= \frac{\partial z_{ca}(n-2)}{\partial v_{rca}(n-2)} = -G \quad \forall n \in \{3,4,\ldots,N\} \\
H(6N+n-2, 2) &= \frac{\partial z_{ab}(n-2)}{\partial \Lambda} = -\frac{\Delta t}{3}(v_{lab}(n) + 4v_{lab}(n-1) + v_{lab}(n-2)) \\
&\forall n \in \{3,4\ldots,N\} \\
H(7N+n-2, 2) &= \frac{\partial z_{bc}(n-2)}{\partial \Lambda} = -\frac{\Delta t}{3}(v_{lbc}(n) + 4v_{lbc}(n-1) + v_{lbc}(n-2)) \\
&\forall n \in \{3,4\ldots,N\} \\
H(8N+n-2, 2) &= \frac{\partial z_{ca}(n-2)}{\partial \Lambda} = -\frac{\Delta t}{3}(v_{lca}(n) + 4v_{lca}(n-1) + v_{lca}(n-2)) \\
&\forall n \in \{3,4\ldots,N\} \\
H(6N+n-2, 2+3N+n) &= \frac{\partial z_{ab}(n-2)}{\partial v_{lab}(n)} = -\frac{\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(7N+n-2, 2+4N+n) &= \frac{\partial z_{bc}(n-2)}{\partial v_{lbc}(n)} = -\frac{\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(8N+n-2, 2+5N+n) &= \frac{\partial z_{ca}(n-2)}{\partial v_{lca}(n)} = -\frac{\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(6N+n-2, 1+3N+n) &= \frac{\partial z_{ab}(n-2)}{\partial v_{lab}(n-1)} = -\frac{4\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(6N+n-2, 1+4N+n) &= \frac{\partial z_{bc}(n-2)}{\partial v_{lbc}(n-1)} = -\frac{4\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(6N+n-2, 1+5N+n) &= \frac{\partial z_{ca}(n-2)}{\partial v_{lca}(n-1)} = -\frac{4\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(6N+n-2, 3N+n) &= \frac{\partial z_{ab}(n-2)}{\partial v_{lab}(n-2)} = -\frac{\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(7N+n-2, 4N+n) &= \frac{\partial z_{bc}(n-2)}{\partial v_{lbc}(n-2)} = -\frac{\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\} \\
H(8N+n-2, 5N+n) &= \frac{\partial z_{ca}(n-2)}{\partial v_{lca}(n-2)} = -\frac{\Delta t\Lambda}{3} \quad \forall n \in \{3,4\ldots,N\}.
\end{aligned}
\tag{96}
$$

The state of the system can then be solved for by applying eqs 14–16.

4 Experiments

Three diferent case-study sytems are considered:

1. A single-phase load
2. a grounded-wye constant-impedance load
3. a delta-connected constant impedance load.

These three load configurations are studied for both phasor state estimation and DSE. In both cases, random noise with an amplitude of approximately 10 % of the signal peak is added to the measurements to verify noise immunity of the methods.

4.1 Phasor Implementation

For the single-phase phasor model, it is assumed that the source voltage is 240 V and the load impedance $R + jX$ is such that it draws a current of $10 - j5$ A. For the three-phase phasor models, both line-ground and line-line fault configurations are considered. These assume that the voltage source is 480 V rms line-line and the load impedance $R+jX$ is such that it draws $30-i15$ A per phase. The fault resistance R_f is selected such that $R_f = R/10$. Measured data is obtained by assuming a balanced input voltage and calculating the current by multipling the input voltage phasor vector by the admittance matrix of the load-fault network. This is also the case for the single-phase dynamic load, though in that case the measured phasor voltage is converted to instantaneous voltage to obtain the input for DSE.

4.2 Dynamic Implementation

The first system is solved ad-hoc assuming an ideal source with the parameters listed in Table 1. The latter two systems are modeled in the MATLAB/Simulink SimScape Specialized Power Systems library with the parameters listed in Tables 2 and 3. In the latter two systems, the load is connected to a 480 V rms line-line source through 1000 ft of 1/0 AWG quadruplex overhead service drop cable. Three different cases are considered:

1. No-fault
2. Line-ground fault
3. Line-line fault.

Table 1: Parameters for Single-Phase Dynamic Load

Variable	Symbol	Value	Units
Total load real power	P	10	kW
Total load reactive power	Q	5	kVAR
Line-line RMS source voltage	V_{ll}	480	V
Simulation time	T	10	ms
Sample rate	T_s	100	μs

Table 2: Common Parameters for Three-Phase Dynamic Models

Variable	Symbol	Value	Units
Total load real power	P	10	kW
Total load reactive power	Q	5	kVAR
Line-line RMS source voltage	V_{ll}	240	V
Source resistance	R_s	19.2	Ω
Source inductance	L_s	25.465	mH
Fault resistance	R_f	1	mΩ
Ground resistance	R_g	10	mΩ
Cable positive-sequence resistance	R_c	183.7	mΩ
Cable positive-sequence reactance	L_c	26.6	mΩ
Simulation time	T	200	ms
Fault start time	T_f	50	ms

Table 3: Varying Parameters for Three-Phase Dynamic Models

Variable	Grounded-Wye	Delta
Load resistance R (Ω)	18.432	55.296
Load inductance L (mH)	24.457	73.3

4.3 Grounded-Wye Load

For the grounded-wye case, the system used in Fig. 13 is used. The load consists of three balanced series RL branches wired in a grounded-wye configuration. This system has the parameters listed in Tables 2 and 3. Note that the total fault resistance for the line-ground fault is $R_f + R_g = 110$ mΩ, while the total fault resistance for the line-line fault is $2R_f = 200$ mΩ.

4.4 Delta Load

For the delta-connected case, the system used in Fig. 14 is used. The load consists of three balanced series RL branches wired in a delta configuration. This system has the parameters listed in Tables 2 and 3. Note that the total

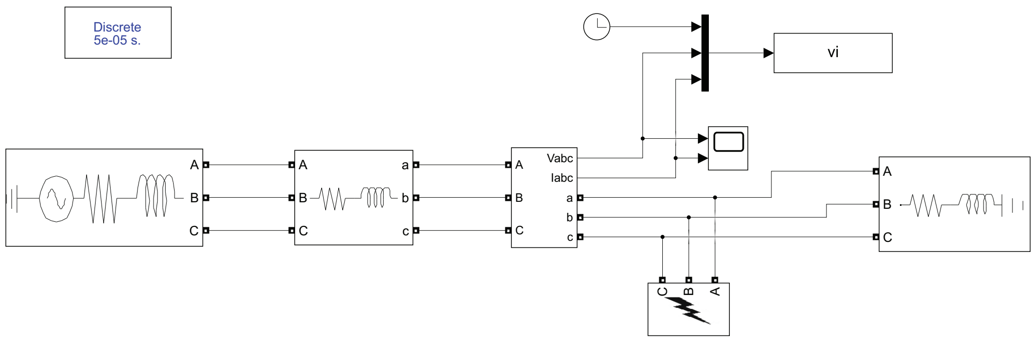


Figure 13: Simulink model for a grounded-wye load with faults

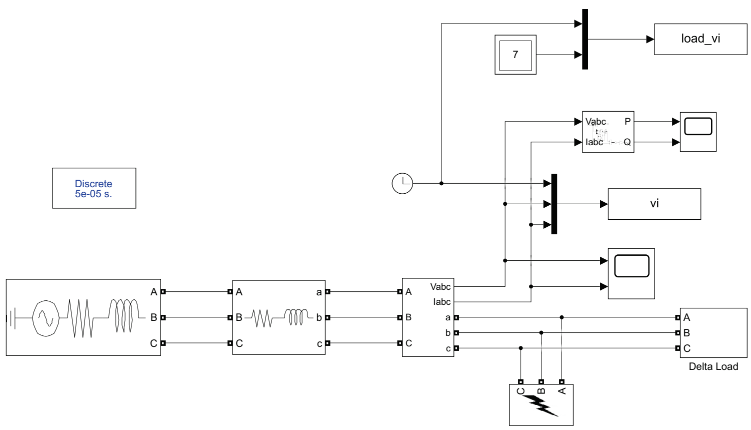
fault resistance for the line-ground fault is $R_f + R_g = 110$ mΩ, while the total fault resistance for the line-line fault is $2R_f = 200$ mΩ.

5 Results

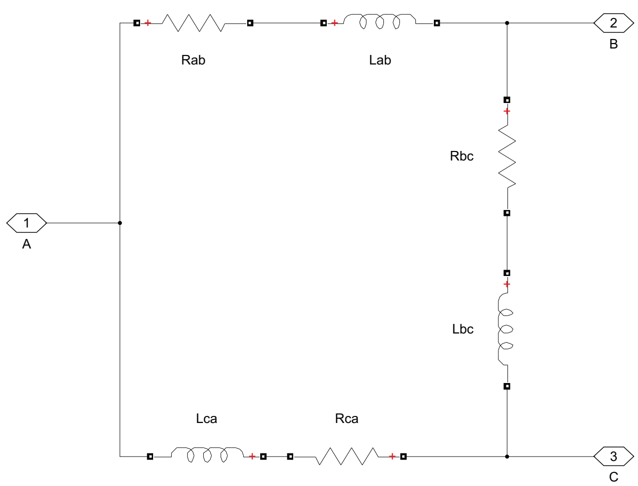
This section presents results of state estimation for the single-phase, grounded-wye and delta load configurations for both the phasor and dynamic cases. Results for the phasor models are presented in Table 4, while results for the dynamic models are presented in Table 5. The phasor models provide an excellent estimate of the system parameters. DSE has difficulty estimating the fault resistance for the dynamic model of grounded-wye network with a line-line fault. A potential solution is to reduce the model order by neglecting the load impedance on the faulted phases. Some moderate error is observed for the case of the delta-connected load with a line-ground fault. Again, it may be possible to improve performance by neglecting load impedance on the faulted phases.

Table 4: Results for Phasor State Estimation

Case	R	$\hat{R}$	X	$\hat{X}$	R_f	$\hat{R}_f$
Single-Phase RL Load	19.200	19.200	9.600	9.600	–	–
Grounded-Wye Line-Ground Fault	7.387	7.387	3.693	3.693	0.923	0.923
Grounded-Wye Line-Line Fault	7.387	5.184	3.693	4.787	0.923	0.935
Delta Line-Line Fault	22.160	22.160	11.080	11.080	2.770	2.770
Delta Line-Ground Fault	22.160	22.160	11.080	11.080	2.770	2.770



(a) Main model



(b) Delta load

Figure 14: Simulink model for a delta load with faults

Table 5: Results for Dynamic State Estimation

Case	R (Ω)	$\hat{R}$ (Ω)	L (mH)	$\hat{L}$ (mH)	R_f ($m\Omega$)	$\hat{R}_f$ ($m\Omega$)
Single-Phase RL Load	19.200	19.265	25.465	25.988	–	–
Grounded-Wye No Fault	18.432	18.404	24.446	24.485	–	–
Grounded-Wye Line-Ground Fault	18.432	18.432	24.446	24.446	11.000	10.997
Grounded-Wye Line-Line Fault	18.432	18.432	24.446	24.446	11.000	3.165
Delta No Fault	55.296	55.412	73.339	73.495	–	–
Delta Line-Line Fault	55.296	55.895	73.339	73.666	2.000	2.001
Delta Line-Ground Fault	55.296	55.479	73.339	73.495	11.000	11.405

6 Conclusions

The results in this study demonstrate that DSE is capable of correctly identifying model parameters of three different load configurations for both normal and faulted operation. These load configurations model a lumped equivalent of a radial electrical network supplying multiple loads. Several models showed sensitivity to inital conditions, particularly the delta-connected load, so it is important that consideration be given to providing the method with good initial conditions. One issue is in making sure that there is a sufficient number of measurements to estimate model states. For example, it is not possible to infer impedances for an unbalanced delta-connected load given a single time snapshot. The models presented here assume that loads are balanced to reduce the number of states. This assumption can be an issue for systems with a high degree of load imbalance

Existing work has demonstrated that DSE can operate with nonlinear elements [13]. One option for future work is to expand the methods here to other load models. These could include nonlinear voltage-dependent models where power is a polynomial function of voltage (ZIP loads) or those where power is a polynomial function of both voltage and frequency such as the WSCC load model [14]. Alternately, these could include dynamic load models such as an induction motor model (MOTORW) or a composite load model (CMPLDW) [15]. Last, there is the possibility of protecting line sections that include loads with coordinated breakers at both ends. This could correspond to a distributed parameter line or a Pi/Tee lumped equivalent model [16].

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7 Appendix

Given a state-output mapping

$$\mathbf{y} - \mathbf{h}(\mathbf{x}) \tag{97}$$

and error-weighting matrix W, the squared error is

$$J = \boldsymbol{\epsilon}' W \boldsymbol{\epsilon}. \tag{98}$$

Given the Jacobian H_0 at $\mathbf{x}_0$, $\mathbf{y}$ can be approximated as

$$\mathbf{y} = \mathbf{h}(\mathbf{x}_0) + H_0(\mathbf{x} - \mathbf{x}_0). \tag{99}$$

Substituting

$$J = [\mathbf{y} - \mathbf{h}(\mathbf{x}_0) - H_0(\mathbf{x} - \mathbf{x}_0)]' W [\mathbf{y} - \mathbf{h}(\mathbf{x}_0) - H_0(\mathbf{x} - \mathbf{x}_0)]. \tag{100}$$

Setting the Jacobian equal to zero

$$\nabla J = -H_0' W [\mathbf{y} - \mathbf{h}(\mathbf{x}_0) - H_0(\mathbf{x} - \mathbf{x}_0)] = \mathbf{0}. \tag{101}$$

Expanding

$$0 = -H_0 W \mathbf{y} + H_0' W H_0 \mathbf{x} - H_0' W H_0 \mathbf{x}_0 - H_0' W \mathbf{h}(\mathbf{x}_0). \tag{102}$$

Rearranging

$$H_0' W H_0 \mathbf{x} = -H_0 W \mathbf{y} + H_0' W H_0 \mathbf{x}_0 + H_0' W \mathbf{h}(\mathbf{x}_0) \tag{103}$$

$$(H_0' W H_0)^{-1} H_0' W H_0 \mathbf{x} = (H_0' W H_0)^{-1} [H_0 W \mathbf{y} + H_0' W H_0 \mathbf{x}_0 + H_0' W \mathbf{h}(\mathbf{x}_0)]. \tag{104}$$

and simplifying

$$H_0' W H_0 \mathbf{x} = -H_0 W \mathbf{y} + H_0' W H_0 \mathbf{x}_0 + H_0' W \mathbf{h}(\mathbf{x}_0) \tag{105}$$

$$\mathbf{x} = (H_0' W H_0)^{-1} [-H_0 W \mathbf{y} + H_0' W H_0 \mathbf{x}_0 + H_0' W \mathbf{h}(\mathbf{x}_0)]. \tag{106}$$

$$\mathbf{x} = (H_0' W H_0)^{-1} [H_0' W H_0 \mathbf{x}_0 + H_0' W \epsilon] \tag{107}$$

$$\mathbf{x} = \mathbf{x}_0 + (H_0' W H_0)^{-1} H_0' W \epsilon. \tag{108}$$

Finally, replacing $\mathbf{x}$ with $\mathbf{x}_{i+1}$, $\mathbf{x}_0$ with $\mathbf{x}_i$ and ϵ with ϵ_i

$$\mathbf{x}_{i+1} = \mathbf{x}_i + (H_0' W H_0)^{-1} H_0' W \epsilon_i. \tag{109}$$

In the case that errors are weighted equally, W is the identity matrix and the update equation simplifies to

$$\mathbf{x}_{i+1} = \mathbf{x}_i + (H_0' H_0)^{-1} H_0' \epsilon_i. \tag{110}$$